A simplified human birth model: Translation of a Rigid Cylinder Through a Passive Elastic Tube

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Motivation

Vaginal delivery is linked to

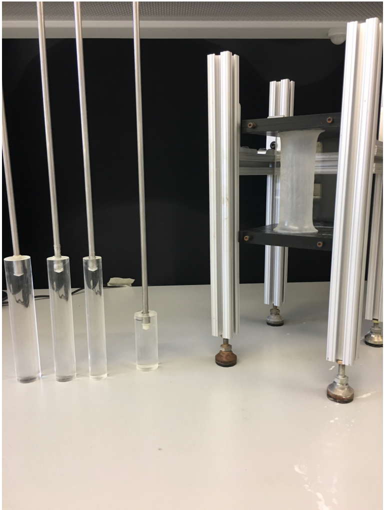
- shorter post-birth hospital stays
- lower likelihood of intensive care stays
- lower mortality rates [1]

Fluid mechanics greatly informs the total mechanics of birth.

- vernix caseosa
- amniotic fluid

[1] C. S. Buhimschi, I. A. Buhimschi (2006). *Advantages of vaginal delivery*, Clinical obstetrics and gynecology.
Fig. 1: "HumanNewborn" by Ernest F - Own work. Licensed under CC BY-SA 3.0 via Commons - `https://commons.wikimedia.org/wiki/File:HumanNewborn.JPG#/media/File:HumanNewborn.JPG`
Fig. 2: "Postpartum baby2" by Tom Adriaenssen - `http://www.flickr.com/photos/inferis/110652572/`. Licensed under CC BY-SA 2.0 via Commons - `https://commons.wikimedia.org/wiki/File:Postpartum_baby2.jpg#/media/File:Postpartum_baby2.jpg`

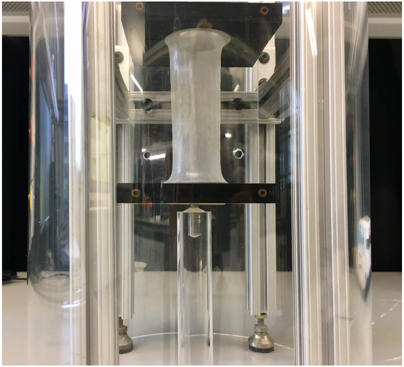
Physical Experiment



Materials:

- Rigid acrylic cylinder (fetus)
- Passive elastic latex tube (birth canal)
- Viscous fluid - methyl cellulose and water (amniotic fluid)

Physical Experiment



The rigid cylinder is pulled at a constant velocity through the center of the elastic tube. The entire system is immersed in fluid.

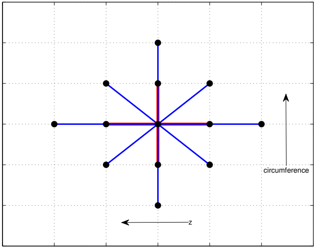
The Model: Solid Behavior

Elastic Tube

- Tube modeled by network of Hookean springs.
- Force at $\mathbf{x}_l$ due to spring from $\mathbf{x}_m$:

$$\mathbf{f}(\mathbf{x}_l) = \tau \left(\frac{\|\mathbf{x}_m - \mathbf{x}_l\|}{\Delta_{lm}} - 1 \right) \frac{(\mathbf{x}_m - \mathbf{x}_l)}{\|\mathbf{x}_m - \mathbf{x}_l\|}$$

where τ =constant, Δ_{lm} =resting distance between points



The Model: Solid Behavior

- Spring constant τ chosen to match elastic properties to physical experiment by setting the total elastic energy of the discrete spring system [2] equal to the stored energy in the continuous elastic tube used in the physical experiment:

$$\sum_{\text{springs}} \frac{\tau}{2\Delta_{lm}} \left(||\mathbf{x}_m - \mathbf{x}_l|| - \Delta_{lm} \right)^2 = \frac{1}{2} A \beta^2 L$$

  where Δ_{lm} =resting distance between points $\mathbf{x}_m, \mathbf{x}_l$, $A = \mathcal{E} I$, $\mathcal{E}$ =Young's modulus of the fluid, I =tube's second moment of area, L =tube length.

[2] H. Nguyen and L. Fauci (2014). *Hydrodynamics of diatom chains and semiflexible fibres*, J. R. Soc. Interface.

The Model: Solid Behavior

Rigid Inner Rod

- A constant velocity **u** is specified in the *z*-direction.

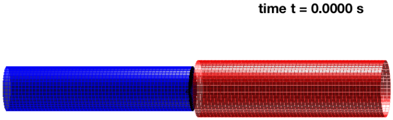


Figure: Discretization of rigid cylinder and elastic tube position in fluid at beginning of simulation.

The Model: Fluid Dynamics

Fluid Behavior is governed by the Stokes equations:

$$
\begin{aligned}
0 &= -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f}, \\
0 &= \nabla \cdot \mathbf{u}
\end{aligned}
$$

For a forces at discrete points $\mathbf{x}_k$ in three dimensions:

$$
0 = -\nabla p + \mu \Delta \mathbf{u} + \sum_{k=0}^{K} \mathbf{f}_0 \delta(\mathbf{x} - \mathbf{x}_k), \nabla \cdot \mathbf{u} = 0
$$

$$
\begin{aligned}
\implies \mathbf{u}(\mathbf{x}) &= \frac{1}{8\pi\mu} \sum_{k=0}^{K} \left[\frac{1}{r_k} \mathbf{f}_k + \frac{(\mathbf{f}_k \cdot \mathbf{x})\mathbf{x}}{r_k^3} \right], \\
p(\mathbf{x}) &= \sum_{k=0}^{K} \mathbf{f}_k \cdot \nabla G(r)
\end{aligned}
$$

where $r_k = ||\mathbf{x}_k - \mathbf{x}||_2, \Delta G(r) = \delta(r)$.

The Model: Fluid Dynamics

The method of regularized stokeslets [3],[4] uses a continuous function ϕ_ε to approximate the dirac-δ function, approximating the Stokes equations for K point forces by

$$0 = -\nabla p + \mu \Delta \mathbf{u} + \textstyle\sum_{k=0}^{K} \mathbf{f}_0 \phi_\varepsilon(\mathbf{x} - \mathbf{x}_k), \nabla \cdot \mathbf{u} = 0,$$

which has solution

$$\mathbf{u}(\mathbf{x}) = \frac{1}{\mu} \sum_{k=1}^{K} \left[(\mathbf{f}_k \cdot \nabla) \nabla B_\varepsilon(|\mathbf{x} - \mathbf{x}_k|) - \mathbf{f}_k G_\varepsilon(|\mathbf{x} - \mathbf{x}_k|) \right],$$

$$p(\mathbf{x}) = \sum_{k=1}^{K} \left[\mathbf{f}_k \cdot \nabla G_\varepsilon(|\mathbf{x} - \mathbf{x}_k|) \right],$$

where $\Delta B_\varepsilon = G_\varepsilon, \Delta G_\varepsilon = \phi_\varepsilon(r) = \frac{15\varepsilon^4}{8\pi(r^2+\varepsilon^2)^{(7/2)}}$.
Here, μ is viscosity, $\mathbf{x}_k$ are points on discretized tube and rod, $\mathbf{f}_k$ is the force at that point, and ε is a regularization parameter.

[3] R. Cortez (2001). *Method of Regularized Stokeslets*, SIAM Journal of Scientific Computing.
[4] R. Cortez, L. Fauci, A. Medovikov (2005). *The method of regularized Stokeslets in three dimensions: analysis, validation, and application to helical swimming*, Physics of Fluids.

The Model: Numerical Solution

Using the solution to the regularized Stokes equations for a given blob function, we can

(1) find the velocity induced on the rod by spring forces in the tube,

(2) solve for any additional forces on the rod necessary to achieve its prescribed velocity,

(3) evaluate the velocity and pressure at every point in the system,

(4) update the tube and rod positions using these velocities one step forward in time.

Choosing a Regularization Parameter: Effect of ε

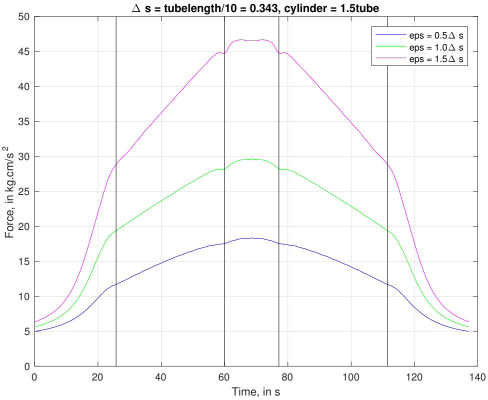


Figure: Computed drag force as a function of time for a rigid cylinder moving through a rigid tube. Force varies greatly with ε.

Drag Force on a Cylinder

Slender body theory provides an asymptotic approximation to the drag force on a long, slender cylinder moving along its axis through a viscous fluid [5]:

$$\mathbf{F} = -\frac{4\pi\mu U L}{2\ln(L/r) - 1}$$

This value can be compared to numerically approximated drag force on a cylinder of fixed radius and varying length, for different values of the regularization parameter ε.

[5] C. Pozrikidis (2011). *Introduction to theoretical and computational fluid dynamics*, Oxford University Press.

Force Error

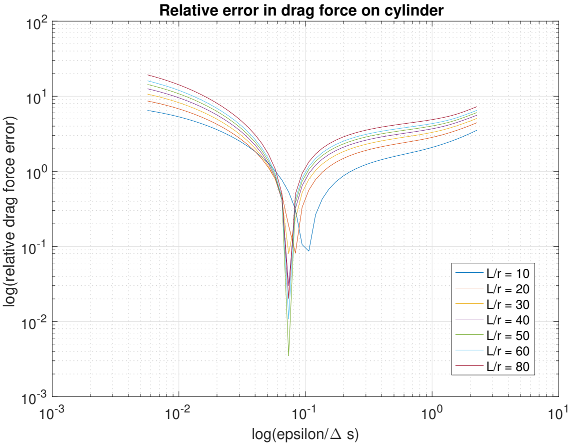


Figure: Error in drag force computation is minimized for ε between $0.07\Delta s$ and $0.08\Delta s$, where Δs is the approximate mesh size of the discretization.

Velocity Error – Leak Test

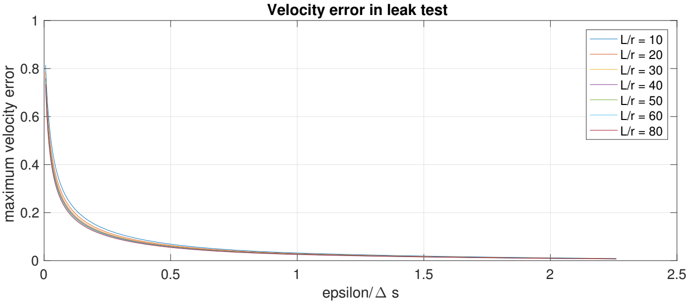


Figure: Measuring the velocity at points on the surface of the cylinder in between the points at which the velocity is specified supplies a measure for how closely the discrete cylinder approximates a solid object. Error decreases as the size of ε increases.

Velocity Profile Between Cylinder and Solid Tube

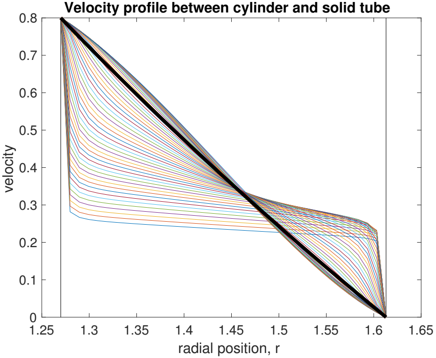


Figure: The velocity profile between an infinitely long rigid cylinder of radius R_c and an infinitely long rigid tube of radius R_t is $u(r) = \frac{U(\ln(R_t) - \ln(r))}{\ln(R_t) - \ln(R_c)}$. This is compared with the velocity computed using the method of regularized stokeslets with varying values for ε.

Velocity Error

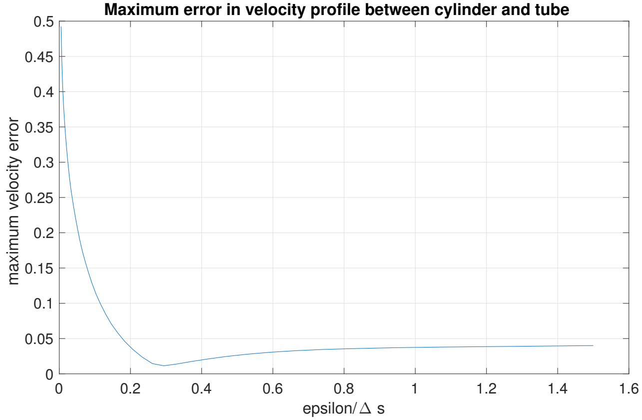
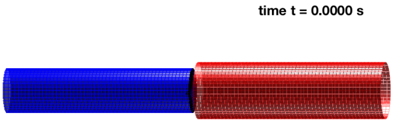


Figure: Error in the velocity profile is minimized for $\varepsilon \approx 0.3\Delta s$.

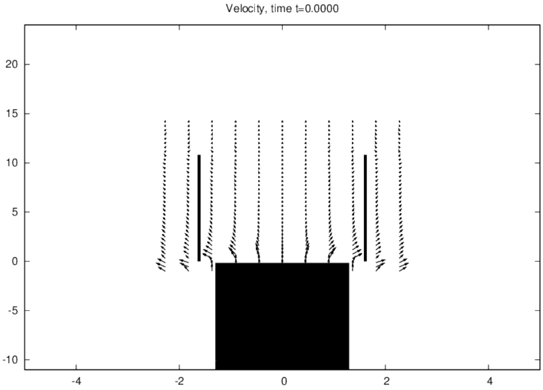
A Simulation

- Tube radius: 1.613 cm
- Cylinder radius: 1.27 cm
- Tube length: 10.8 cm
- Cylinder length: 13.2 cm
- Fluid viscosity: 0.0305 Pa·s
- Elastic Young's modulus: 0.01GPa
- Regularization parameter: $\varepsilon = 0.5\Delta s$

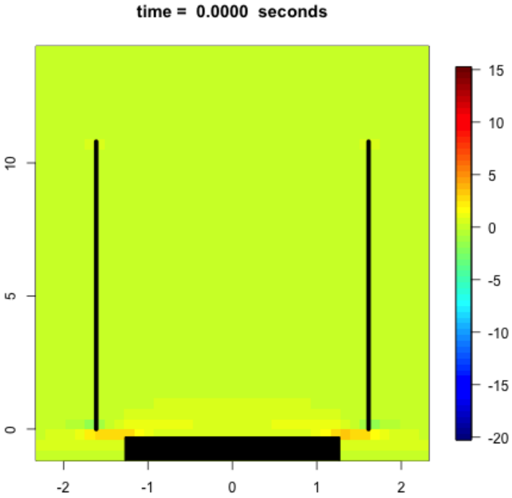
System Behavior



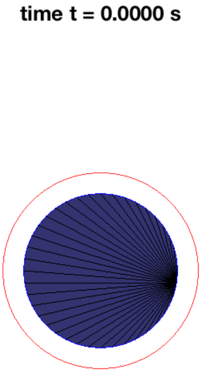
Fluid Velocity



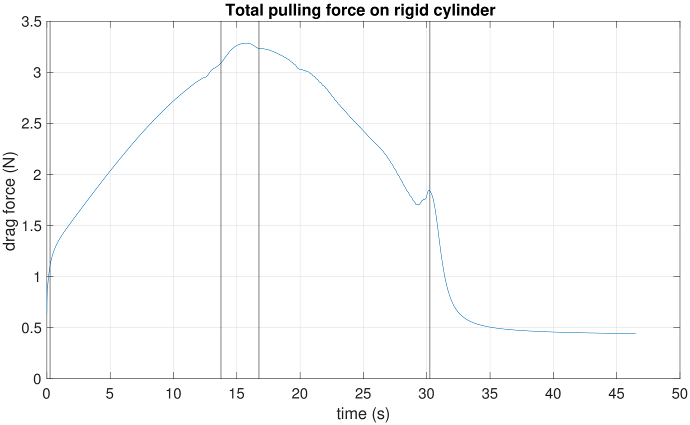
Fluid Pressure



Tube Buckling



Pulling Force



Future Work

- Further analysis of tube buckling behavior
  - Different diameter and length of rigid cylinder
  - Different velocity of rigid cylinder
  - Different elasticity of tube
- Find the “perfect” regularization parameter value
- Increase realism
  - active elastic tube / modeling peristalsis
  - more accurate geometry

Slides available at
`math.tulane.edu/~rpealate`