A simplified human birth model: translation of a rigid cylinder through a passive elastic tube

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Motivation

Vaginal delivery is linked to

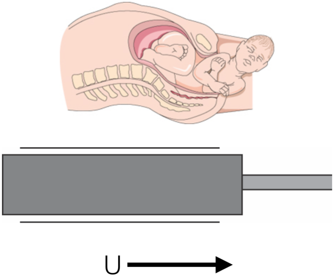
- shorter post-birth hospital stays
- lower likelihood of intensive care stays
- lower mortality rates [1]

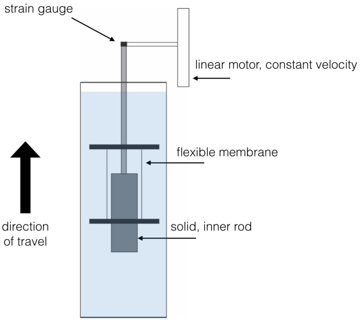
Fluid mechanics greatly informs the total mechanics of birth. [2]

- How do the involved fluids inform the forces on an infant during birth?

[1] C. S. Buhimschi, I. A. Buhimschi (2006). *Advantages of vaginal delivery*, Clinical obstetrics and gynecology.
[2] A. M. Lehn, A. Baumer, M. C. Leftwich, *An experimental approach to a simplified model of human birth*.
Fig. 1: "HumanNewborn" by Ernest F - Own work. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:HumanNewborn.JPG#/media/File:HumanNewborn.JPG
Fig. 2: "Postpartum baby2" by Tom Adriaenssen - http://www.flickr.com/photos/inferis/110652572/. Licensed under CC BY-SA 2.0 via Commons - https://commons.wikimedia.org/wiki/File:Postpartum_baby2.jpg#/media/File:Postpartum_baby2.jpg

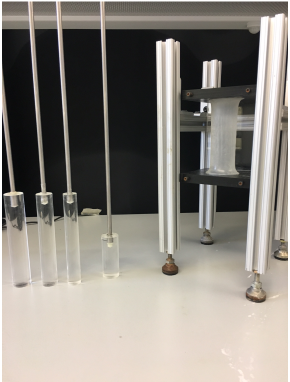
A simplified model

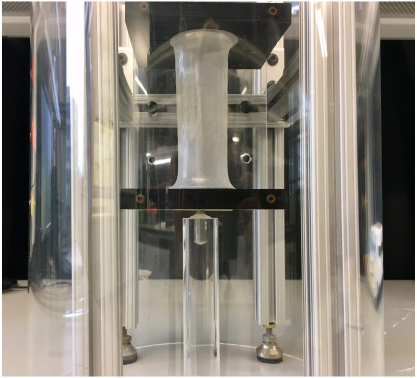




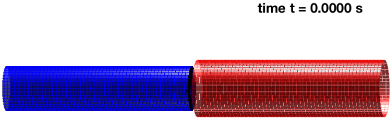
- Rigid acrylic cylinder (fetus)
- Passive elastic tube (birth canal)
- Viscous fluid (amniotic fluid)
- Rigid cylinder is pulled through center of elastic tube at prescribed velocity, and drag force is measured.

Physical Experiment





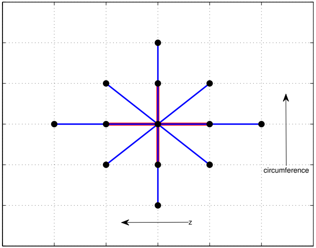
Numerical Model



Numerical model: solid behavior

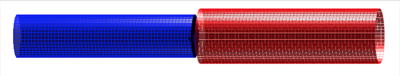
Elastic tube

- Tube modeled by network of Hookean springs.
- Force at point $\mathbf{x}_l$ due to spring from point $\mathbf{x}_m$:
  $\mathbf{g}(\mathbf{x}_l) = \tau \left(\frac{\|\mathbf{x}_m - \mathbf{x}_l\|}{\Delta_{lm}} - 1 \right) \frac{(\mathbf{x}_m - \mathbf{x}_l)}{\|\mathbf{x}_m - \mathbf{x}_l\|}$
- τ chosen to match elastic properties to physical experiment. [3]



Rigid inner cylinder

- A constant velocity $\mathbf{u} = U$ is specified in the z-direction.



[3] H. Nguyen and L. Fauci (2014). *Hydrodynamics of diatom chains and semiflexible fibres*, J. R. Soc. Interface.

Numerical model: fluid dynamics

Fluid Behavior is governed by the Stokes equations, with regularized forces at K discrete points in the system:

$$0 = -\nabla p + \mu \Delta \mathbf{u} + \sum_{k=0}^{K} \mathbf{f}_k \phi_\varepsilon(\mathbf{x} - \mathbf{x}_k), \nabla \cdot \mathbf{u} = 0,$$

which have solution [4],[5]

$$\mathbf{u}(\mathbf{x}) = \frac{1}{\mu} \sum_{k=1}^{K} \left[(\mathbf{f}_k \cdot \nabla) \nabla B_\varepsilon(|\mathbf{x} - \mathbf{x}_k|) - \mathbf{f}_k G_\varepsilon(|\mathbf{x} - \mathbf{x}_k|) \right],$$

$$p(\mathbf{x}) = \sum_{k=1}^{K} \left[\mathbf{f}_k \cdot \nabla G_\varepsilon(|\mathbf{x} - \mathbf{x}_k|) \right],$$

where $\Delta B_\varepsilon = G_\varepsilon, \Delta G_\varepsilon = \phi_\varepsilon(r) = \frac{15\varepsilon^4}{8\pi(r^2+\varepsilon^2)^{(7/2)}}$.
Here, μ is viscosity, $\mathbf{x}_k$ are points on discretized tube and rod, $\mathbf{f}_k$ is the force at that point, and ε is a regularization parameter.

[4] R. Cortez (2001). *Method of Regularized Stokeslets*, SIAM Journal of Scientific Computing.
[5] R. Cortez, L. Fauci, A. Medovikov (2005). *The method of regularized Stokeslets in three dimensions: analysis, validation, and application to helical swimming*, Physics of Fluids.

Numerical model: algorithm

Using the solution to the regularized Stokes equations for a given regularization function ϕ_ε:

(1) Calculate spring forces in the tube based on its deformation, and calculate the velocity they induce on the inner cylinder.

(2) Solve for additional forces necessary on inner cylinder points to achieve its desired constant velocity.

(3) Evaluate the velocity points on tube and inner cylinder. (Velocity and pressure can be evaluated at any other point in the system.)

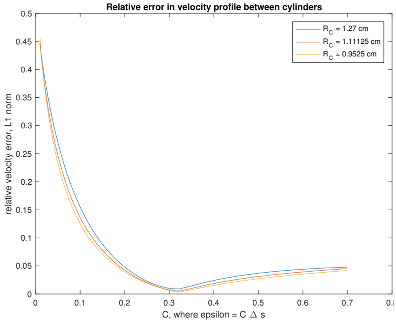
(4) Update the tube and rod positions using these velocities one step forward in time.

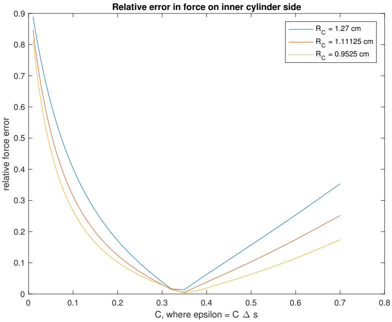
(5) Repeat.

Validation: concentric rigid cylinders

For concentric rigid cylinders of infinite length, with outer tube of radius R_T fixed and inner cylinder of radius R_C moving at constant velocity U:

- Velocity profile between cylinders is given by: $u(r) = \frac{U(\ln(R_t) - \ln(r))}{\ln(R_t) - \ln(R_c)}$
- Traction at a point on the side of inner cylinder is: $t = \frac{\mu U}{R_c \ln\left(\frac{R_T}{R_C}\right)}$
- Compared to numerical results for finite-length concentric rigid cylinders:





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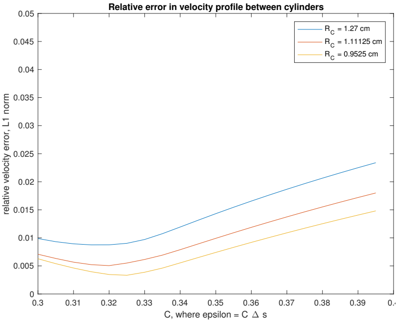
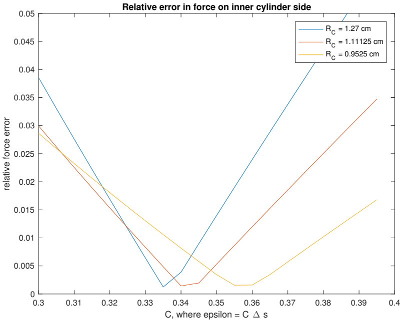
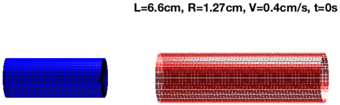




A representative simulation



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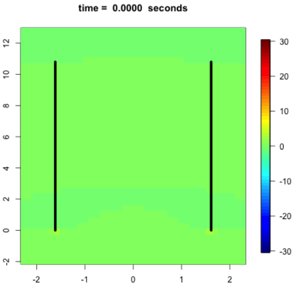
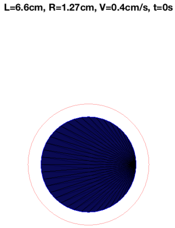
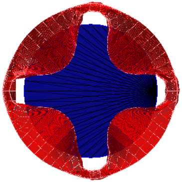


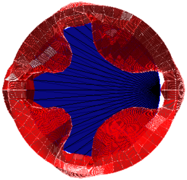


Tube buckling

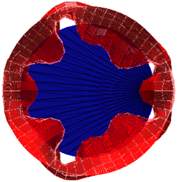
$U = 0.4$



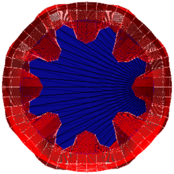
$U = 0.8$



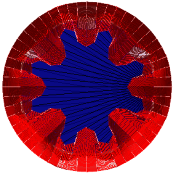
$U = 1.6$



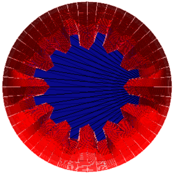
$U = 3.2$



$U = 6.4$



$U = 12.8$



An elastohydrodynamic ratio for invariant elastic behavior

The dimensionless ratio of fluid and elastic "forces"

$$\frac{\mu U}{L\mathcal{E}}$$

arises from nondimensionalizing the Stokes equations, where

$$\begin{aligned} \mu &= \text{ fluid viscosity,} \\ U &= \text{ fluid velocity,} \\ \mathcal{E} &= \text{ Young's modulus of tube,} \\ I &= \text{ second moment of area of tube,} \\ L &= \text{ characteristic length scale} \end{aligned}$$

An elastohydrodynamic ratio for invariant elastic behavior

The dimensionless ratio of fluid and elastic “forces”

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arises from nondimensionalizing the Stokes equations

ROAD BLOCK: There is no single characteristic length scale in this problem. The elastic behavior varies with tube length, tube radius, inner cylinder length, inner cylinder radius (or equivalently, gap size), as well as mesh choice.

An elastohydrodynamic ratio for invariant elastic behavior

The dimensionless ratio of fluid and elastic “forces”

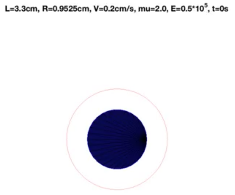
$$\frac{\mu U}{L\mathcal{E}}$$

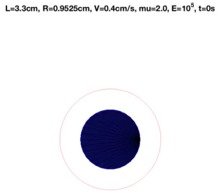
arises from nondimensionalizing the Stokes equations

SOLUTION: Keeping the geometry constant, we consider the dimensional elastohydrodynamic ratio

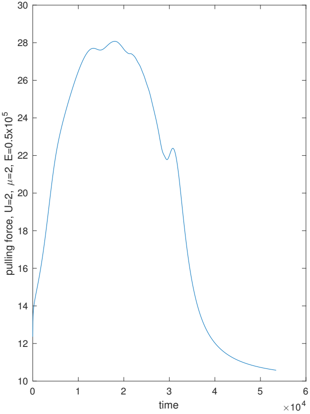
$$\frac{\mu U}{\mathcal{E}}$$

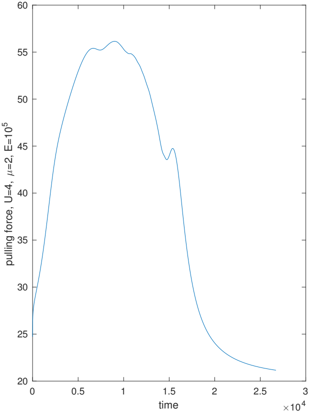
Tube buckling under constant elastohydrodynamic ratio





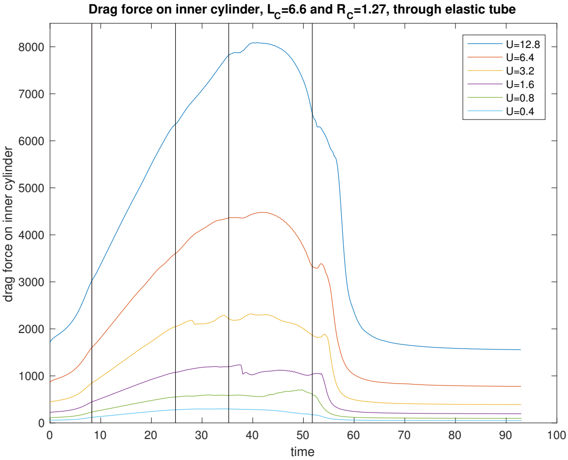
Pulling force under constant elastohydrodynamic ratio





Note that the relationship between force and velocity (and between force and viscosity) is linear, as long as this ratio is constant.

Pulling force for varying elastohydrodynamic ratio



Future work

- Comparison to physical experiment results
- Increase of realism
    - Model peristalsis...
        - ...through prescribed periodic velocity
        - ...through prescribed periodic forcing
        - ...through active elastic tube undergoing periodic contractions
    - Explore effect of lateral displacement of inner cylinder
    - Vary elastic properties throughout surface of tube
    - Add more accurate geometry

Slides available at
`math.tulane.edu/~rpealate`