What do I do if there is no (analytical) solution???

An introduction to numerical differential equations

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Importance of Numerical Methods

Example 1: The Standard Normal Distribution

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1$$

But

$$P(X < c) = \int_{-\infty}^{c} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = ???$$

Importance of Numerical Methods

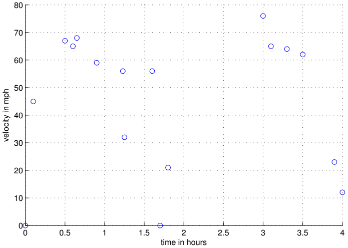
Example 2: Unknown Function Values

A car drives along a set route. At random intervals during that time, we know the car's speed. How can we find the function defining the car's position (measured in distance from its start) as a function of time?

In other words, we want to solve

$$\frac{dx}{dt} = v(t)$$

for the position $x(t)$, but the information we know is something like:



Importance of Numerical Methods

Example 3: The Navier-Stokes Equations

$$\rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) = -\nabla p + (\mu + \lambda)\nabla(\nabla \cdot \mathbf{u}) + \mu\Delta\mathbf{u} + \mathbf{f}$$

Want a million dollars?

Necessary Background

Recall the following definition of the derivative of a function $f(x)$ at the point x_0:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x}$$

Necessary Background

Recall the Taylor series expansion of a function $f(x)$ about the point x_0:

$$
\begin{aligned}
f(x) =& f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x0)^2 f''(x_0) + ... \\
& ... + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0) + ...
\end{aligned}
$$

Today's Problem

Given the first-order initial value problem

$$y'(t) = f(y(t), t), y(0) = y_0$$

we construct a numerical method

$$u_{n+1} = \phi(u_n, h), u_0 = y_0$$

where h is the time-step size.

What do we want from a numerical method?

A good numerical method is Efficient and Effective.
Measuring the "correctness" of a numerical method:

$$\text{Consistency} + \text{Stability} = \text{Convergence}$$

What do we want from a numerical method?

A numerical method is *stable* if there exists a constant C such that, for all n,

$$|u_n| \leq C|u_0|$$

A numerical method is *consistent* if

$$\tau \rightarrow 0 \text{ as h } \rightarrow 0$$

where the truncation error τ is the remainder when the true solution is substituted into the numerical method.

Forward Euler

$$u_{n+1} = u_n + hf_n$$

Stop! Example time!

Backward Euler

$$u_{n+1} = u_n + hf_{n+1}$$

Same order of convergence as backward Euler, but implicit, and thus more stable.

Crank-Nicolson

$$u_{n+1} = u_n + h(f_n + f_{n+1})$$

Recognize the trapezoid rule?
Stop! Example time again! (Well, maybe.)

Heun's Method

But using implicit methods is not as efficient, so...

$$u_{n+1} = u_n + h(f_n + f(t_{n+1}, u_n + hf_n))$$

This is a predictor-corrector method with Forward Euler as a predictor, and Crank-Nicolson as a corrector. Heun's method has the same order of convergence as Crank-Nicolson but since it is explicit, it saves on computation time.

How do you come up with these?

Well, here's one way....
Recall the definition of the derivative of y at t given by:

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$$

If we use the RHS as an approximation to the derivative, and substitute into the ODE $y'(t) = f(y,t)$, we get

$$\frac{y(t_{n+1}) - y(t_n)}{h} = f_n$$

Rearranging terms gives us forward Euler.
This is one example of a finite difference method.

Finite Differences

Stop! Example time!

So what do I do?

Motivation

Vaginal delivery is linked to

- shorter post-birth hospital stays
- lower likelihood of intensive care stays
- lower mortality rates [1]

Fluid mechanics greatly informs the total mechanics of birth.

- vernix caseosa
- amniotic fluid

[1] C. S. Buhimschi, I. A. Buhimschi (2006). *Advantages of vaginal delivery*, Clinical obstetrics and gynecology.

The Model: Solid Behavior

Elastic Tube (Super Simplified Birth Canal)

- Tube modeled by network of Hookean springs.
- Force at $\mathbf{x}_l$ due to spring from $\mathbf{x}_m$:
  $\mathbf{f}(\mathbf{x}_l) = \tau \left(\frac{\|\mathbf{x}_m - \mathbf{x}_l\|}{\Delta_{lm}} - 1 \right) \frac{(\mathbf{x}_m - \mathbf{x}_l)}{\|\mathbf{x}_m - \mathbf{x}_l\|}$
- τ chosen to match elastic properties to physical experiment. [2]

Rigid Inner Cylinder (Super Simplified Baby)

- A constant velocity $\mathbf{u}$ is specified in the z-direction.

[2] H. Nguyen and L. Fauci (2014). *Hydrodynamics of diatom chains and semiflexible fibres*, J. R. Soc. Interface.

The Model: Fluid Dynamics

Fluid Behavior is governed by the Stokes equations:

$$0 = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f},$$

$$\nabla \cdot \mathbf{u} = 0.$$

The linear relationship between fluid velocities and regularized forces localized at N points is given by

$$\mathbf{u}(\mathbf{x}) = \frac{1}{\mu} \sum_{k=1}^{K} \left[(\mathbf{f}_k \cdot \nabla) \nabla B_\varepsilon(|\mathbf{x} - \mathbf{x}_k|) - \mathbf{f}_k G_\varepsilon(|\mathbf{x} - \mathbf{x}_k|) + \mathbf{u}_b(\mathbf{x}) \right],$$
$$p(\mathbf{x}) = \sum_{k=1}^{K} \left[\mathbf{f}_k \cdot \nabla G_\varepsilon(|\mathbf{x} - \mathbf{x}_k|) \right],$$

where $\Delta B_\varepsilon = G_\varepsilon, \Delta G_\varepsilon = \phi_\varepsilon, \phi_\varepsilon(r) = \frac{15\varepsilon^4}{8\pi(r^2+\varepsilon^2)^{(7/2)}}$

Here, μ is viscosity, $\mathbf{x}_k$ are points on discretized tube and rod, $\mathbf{f}_k$ is the force at that point, and ε is a regularization parameter. [3],[4]

[3] R. Cortez (2001). *Method of Regularized Stokeslets*, SIAM Journal of Scientific Computing.
[4] R. Cortez, L. Fauci, A. Medovikov (2005). *The method of regularized Stokeslets in three dimensions: analysis, validation, and application to helical swimming*, Physics of Fluids.

The Model: Numerical Solution

Using the solution to the regularized Stokes equations for a given blob function, we can

(1) find the velocity induced on the rod by spring forces in the tube,

(2) solve for any additional forces on the rod necessary to achieve its prescribed velocity,

(3) evaluate the velocity and pressure at every point in the system,

(4) update the tube and rod positions using these velocities one step forward in time.

System Behavior