The elastohydrodynamics of a simplified model of human birth

Roseanna Gossmann[1], Alexa Baumer[2], Lisa Fauci[1], Megan C. Leftwich[2]

Tulane University[1], George Washington University[2]

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The Human Birth Problem

Maternal mortality and morbidity in the US [1]

- higher than most other "high-income" countries
- increasing over time

Caesarean delivery

- 32% of all deliveries in the US [2]
- WHO recommends 10-15% [3]
- linked to higher mortality [4] and morbidity [5] rates

[1] *Bulletin of the World Health Organization*, 2015
[2] *CDC National Center for Health Statistics*, 2015
[3] *WHO Statement on Caesarean Section Rates*, 2015
[4] Buhimschi, Buhimschi, *Clinical Obstetrics and Gynecology*, 2006
[5] Burrows, Meyn, Weber, *Obstetrics&Gynecology*, 2004

Fig. 1: "HumanNewborn" by Ernest F - Own work. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:HumanNewborn.JPG#/media/File:HumanNewborn.JPG
Fig. 2: "Postpartum baby2" by Tom Adriaenssen - http://www.flickr.com/photos/inferis/110652572/. Licensed under CC BY-SA 2.0 via Commons - https://commons.wikimedia.org/wiki/File:Postpartum_baby2.jpg#/media/File:Postpartum_baby2.jpg

Motivation

A better understanding of the mechanics of human birth may decrease the incidence of unnecessary surgical delivery.

Considering Fluid Dynamics Is Essential

Amniotic fluid is highly variable

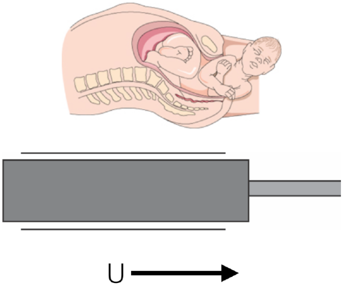
- in volume [6]
- in rheological properties [7]

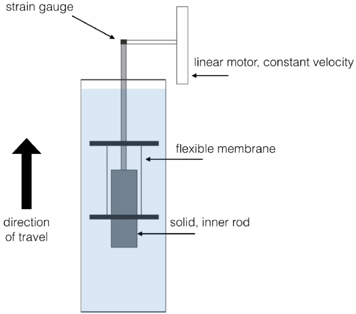
It is unknown how these fluid properties affect the transfer of force from the uterus onto the baby during delivery.

Fluid dynamics were shown to significantly affect the force necessary for delivery in a model vacuum-assisted delivery. [8]

[6] Brace, Wolf, *American J. of Obstetrics and Gynecology*, 1989
[7] Uyeno, *J. of Biological Chemistry*, 1919
[8] Lehn, Baumer, Leftwich, *J. of Biomechanics*, 2016

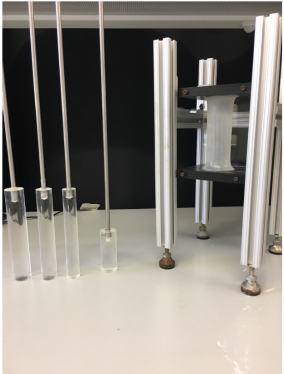
A Simplified Physical Experiment

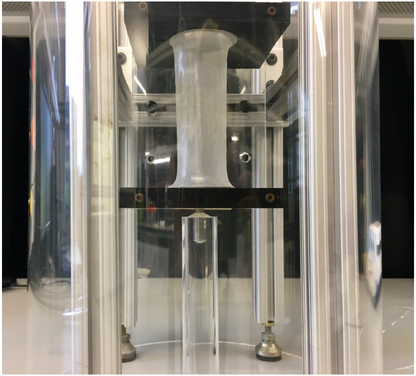




Schematics courtesy of Alexa Baumer, Leftwich laboratory, The George Washington University

A Simplified Physical Experiment

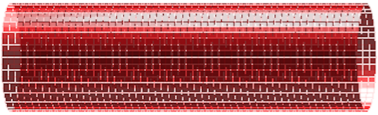


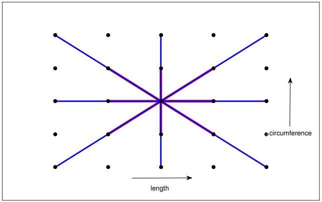


Physical experiment at Leftwich laboratory, The George Washington University

Spring Network Model of Elastic Tube

- Tube modeled by network of Hookean springs oriented tangentially, circumferentially, and helically.
- Force at point $\mathbf{x}_l$ due to spring from point $\mathbf{x}_m$:
  $\mathbf{g}(\mathbf{x}_l) = \tau \left(\frac{\|\mathbf{x}_m - \mathbf{x}_l\|}{\Delta_{lm}} - 1 \right) \frac{(\mathbf{x}_m - \mathbf{x}_l)}{\|\mathbf{x}_m - \mathbf{x}_l\|}$
- Total force due to springs at the points $\mathbf{x}_l$ is the sum of forces from 10 to 16 springs connected to $\mathbf{x}_l$.





Spring Network Model of Elastic Tube

Total elastic energy stored in the discrete spring system:

$$E_n = \sum_{\text{springs}} \frac{\tau}{2\Delta_{lm}} (||\mathbf{x}_m - \mathbf{x}_l|| - \Delta_{lm})^2$$

where τ is the spring constant of every spring, Δ_{lm} is the spring resting length.

Total elastic energy stored in a homogeneous elastic tube:

$$E = \frac{1}{2} A \beta^2 L$$

where $A = \mathcal{E} I$ is the bending stiffness of the tube, $\mathcal{E}$ is the tube's Young's modulus, I is the tube's second moment of area, β its curvature, and L its length.

By enforcing $E_n = E$ for various curvatures β, we can relate individual spring stiffness to macroscopic elastic energy to choose τ.

Method of Regularized Stokeslets

Stokes equations for regularized forces:

$$0 = \mu \Delta \mathbf{u} - \nabla p + \mathbf{f}_0 \phi_\varepsilon(\mathbf{x} - \mathbf{x}_0)$$
$$0 = \nabla \cdot \mathbf{u}$$

Solution to regularized Stokes equations:

$$\mathbf{u}(\mathbf{x}) = \frac{1}{\mu}(\mathbf{f}_0 \cdot \nabla)\nabla B\varepsilon - \mathbf{f}_0 G_\varepsilon$$

where $\Delta G_\varepsilon = \phi_\varepsilon(\mathbf{x} - \mathbf{x}_0), \Delta B_\varepsilon = G_\varepsilon$.

We use the blob function $\phi_\varepsilon = \frac{15\varepsilon^4}{8\pi(||\mathbf{x} - \mathbf{x}_0||^2 + \varepsilon^2)^{(7/2)}}$. [9]

[9] Cortez, Fauci, Medovikov, *Physics of Fluids*, 2005

Linearity of Stokes Equations

Since the Stokes equations are linear, for regularized forces at points $\mathbf{x}_n$,

$$0 = \mu\Delta\mathbf{u} - \nabla p + \sum_n \mathbf{f}_n \phi_\varepsilon(\mathbf{x} - \mathbf{x}_n)$$

$$0 = \nabla \cdot \mathbf{u}$$

has solution

$$\mathbf{u}(\mathbf{x}) = \sum_n \frac{1}{\mu}(\mathbf{f}_n \cdot \nabla)\nabla B\varepsilon - \mathbf{f}_n G_\varepsilon$$

Algorithm

Using the solution to the regularized Stokes equations $\mathbf{u} = A\mathbf{f}$

(1) Calculate spring forces in the tube based on its deformation. Calculate the velocity they induce on the inner cylinder and fixed tube ends (matrix multiplication).

(2) Solve for additional forces necessary on inner cylinder and tube ends to achieve prescribed velocities, using BiCGSTAB iterative method to solve linear system. [10]

(3) Evaluate the velocity at points on tube (matrix multiplication).

(4) Update the tube and rod positions using these velocities and prescribed velocities one step forward in time, using Forward Euler time-stepping.

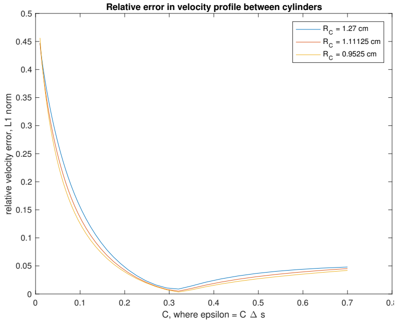
(5) Repeat.

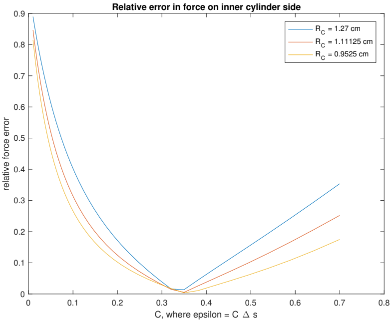
[10] Van der Vorst, *SIAM J. Sci. Stat. Comput.*, 1992

Model Validation and Regularization Parameter Choice

For concentric rigid cylinders of infinite length, with outer tube of radius R_T fixed and inner cylinder of radius R_C moving at constant velocity U:

- Velocity profile between cylinders is given by: $u(r) = \frac{U(\ln(R_t) - \ln(r))}{\ln(R_t) - \ln(R_c)}$
- Traction at a point on the side of inner cylinder is: $t = \frac{\mu U}{R_c \ln\left(\frac{R_T}{R_C}\right)}$
- Compared to numerical results for finite-length concentric rigid cylinders:





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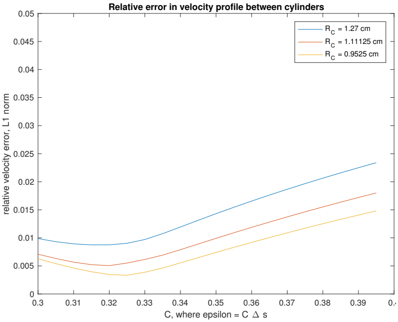
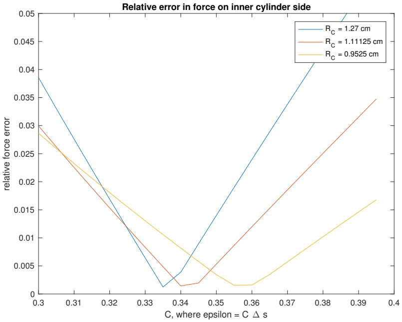
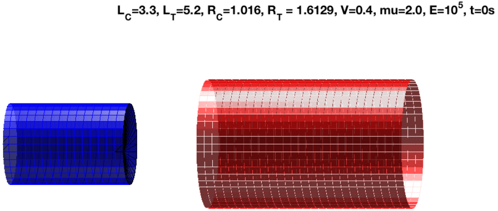


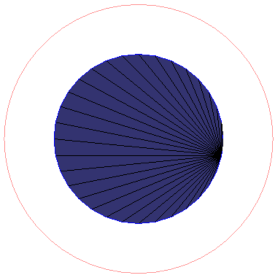


Results

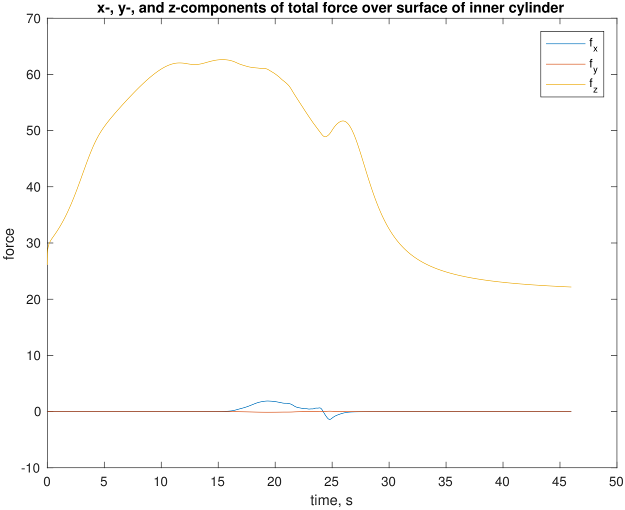


Results

L_C=3.3, L_T=5.2, R_C=1.016, R_T = 1.6129, V=0.4, mu=2.0, E=10^5, t=0s



Results



x-, y-, and z-components of total force over surface of inner cylinder
force
time, s
70
60
50
40
30
20
10
0
-10
0
5
10
15
20
25
30
35
40
45
50
f_x
f_y
f_z

Model Modification - Elastic Inner Cylinder

In order to improve two issues

- solving the linear system is computationally expensive
- force input is more realistic for biological applications than prescribed velocity

we build an elastic spring cylinder, with every point connected to every other point by a spring with force

$$\mathbf{g}_c(\mathbf{x}_l) = \tau_c \left(\frac{\|\mathbf{x}_m - \mathbf{x}_l\|}{\Delta_{lm}} - 1 \right) \frac{(\mathbf{x}_m - \mathbf{x}_l)}{\|\mathbf{x}_m - \mathbf{x}_l\|}$$

and we anchor the tube ends with forces that penalize moving away from their initial positions

$$\mathbf{g}_p(\mathbf{x}_l) = \tau_p||\mathbf{x}_m - \mathbf{x}_l||(\mathbf{x}_m - \mathbf{x}_l)$$

Model Modification - Elastic Inner Cylinder

Using the solution to the regularized Stokes equations $\mathbf{u} = A\mathbf{f}$

(1) Calculate spring forces in the tube and in the inner cylinder based on their deformation.

(2) Calculate penalty forces on tube ends based on their position.

(3) Add prescribed forces to inner cylinder to "push" it through the tube.

(3) Evaluate the velocity at points on tube and inner cylinder (matrix multiplication).

(4) Update the tube and rod positions using these velocities and prescribed velocities one step forward in time, using Forward Euler time-stepping.

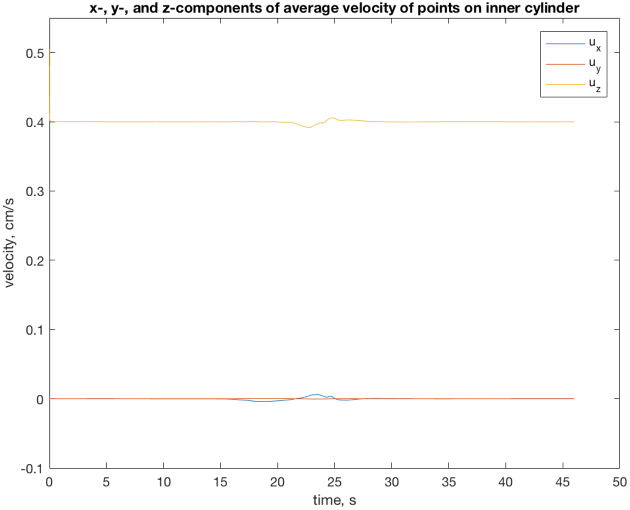
(5) Repeat.

Model Modification - Elastic Inner Cylinder

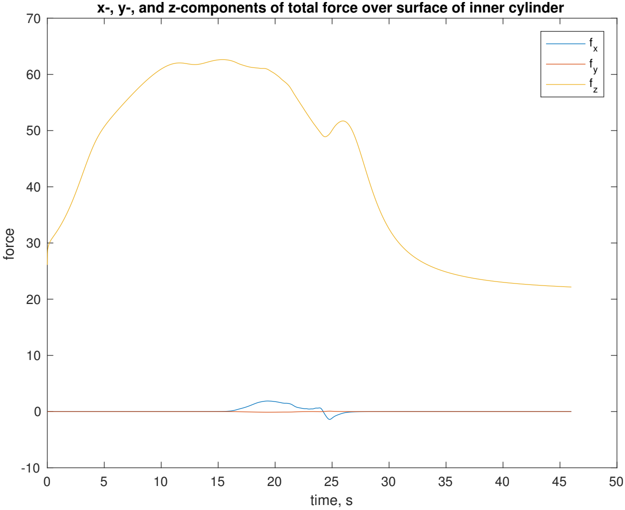
Benefits

- speed
  - Method #1: BiCGSTAB + $2[A\mathbf{f}] = O(Nn^3)$ flops, where N = number of iterations to convergence (≈ 100), n = number of rows in A
  - Method #2: $[A\mathbf{f}] = O(n^3)$ flops, also linear speed-up
  - Small sample problem ($n = 6396$, assume $N = 100$):
    - 1.84×10^{14} vs. 2.62×10^{11}
- any force input

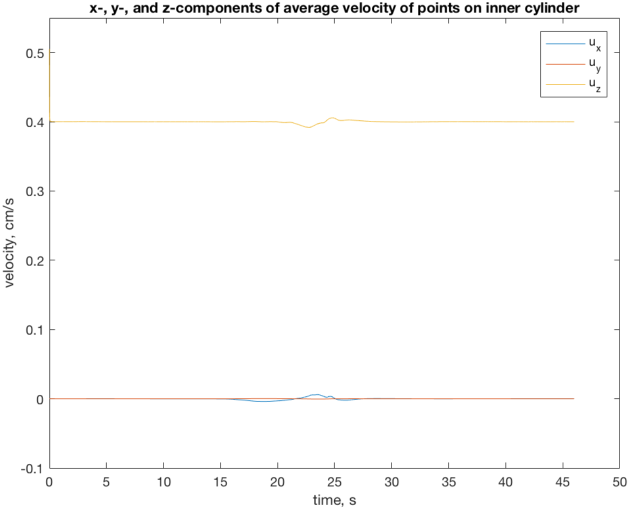
Results


x-, y-, and z-components of average velocity of points on inner cylinder
u_x
u_y
u_z
velocity, cm/s
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
5
10
15
20
25
30
35
40
45
50
time, s

Force Input


x-, y-, and z-components of total force over surface of inner cylinder
70
60
50
40
30
20
10
0
-10
force
0
5
10
15
20
25
30
35
40
45
50
time, s
f
x
f
y
f
z

Results



x-, y-, and z-components of average velocity of points on inner cylinder
velocity, cm/s
time, s
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
5
10
15
20
25
30
35
40
45
50
u_x
u_y
u_z