Tulane University

Department Of Mathematics

Applied Mathematics 1 & 2 Summary

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Contents

Part I

Applied Mathematics 1

1 Dimensional Analysis

Dimensions of Common Physical Quantities

$[L]$	Length	L
$[M]$	Mass	M
$[T]$	Time	T
$[v]$	Velocity	LT^{-1}
$[a]$	Acceleration	LT^{-2}
$[F]$	Force	MLT^{-2}
$[\rho]$	Mass density	ML^{-3}
$[p]$	Pressure	$ML^{-1}T^{-2}$
$[E]$	Energy	ML^2T^{-2}

Buckingham Pi Theorem

When a complete relationship between dimensional physical quantities is expressed in dimensionless form, the number of independent quantities that appear in it is reduced from the original n to $n-k$, where k is the maximum number of the original n that are dimensionally independent.

Step 1: Identify a complete set of relevant physically independent variables $Q_1, Q_2, \cdots, Q_n$, and their dimensions.
Step 2: Identify a complete, dimensionally independent subset $Q_1, \cdots, Q_k$ where $Q_{k+1}, \cdots, Q_n$ have dimensions that can be expressed as products of powers of the dimensions of the first k variables.
Step 3: Find $n-k$ dimensionless quantities by solving matrix equations as shown in examples to follow.
Step 4: Each dimensionless quantity can be written as a function of the others.

Examples

See Homework 2

Exam Problems

Problem 1. Consider the force, F, of the wind (with density ρ, traveling at speed v, and absolute/dynamic viscosity μ) blowing against a barge. Suppose that the barge has length l, height h, and width w. Use the labels M for mass, L for length, T for time. For example, the viscosity has dimensions $[\mu] = ML^{-1}T^{-1}$.

(a) What are the dimensionless variables. State and apply the Buckingham Pi theorem to determine how the force of the wind on the barge depends upon the wind velocity.

(b) Suppose that the barge is five times longer than it is wide, $l = 5w$, and ten times longer than it is high, $l = 10h$. What is the difference in the force of the wind on the front versus the side of the barge.

Solution.
(a)
We consider the following variables with their corresponding physical dimensions:

$$[F] = MLT^{-2}, [\rho] = ML^{-3}, [v] = LT^{-1}, [\mu] = ML^{-1}T^{-1}, [A] = L^2$$

(A denotes area, other variables are as in the problem description above).
In order to find all dimensionless quantities we solve for the null space of the following matrix:

	A	ρ	μ	v	F
L	2	-3	-1	1	1
M	0	1	1	0	1
T	0	0	-1	-1	-2

We find that the null space is spanned by the vectors $(-1/2, 0, -1, -1, 1)^T$ and $(1/2, 1, -1, 1, 0)^T$. Thus, a complete and independent set of dimensionless quantities is,

$$\Pi_1 = \frac{F}{v\mu\sqrt{A}},$$
$$\Pi_2 = \frac{\sqrt{A}\rho v}{\mu}.$$

Theorem (Buckingham Pi Theorem)**.** *Suppose that we have a relationship between a quantity a which is being determined in some experiment, and a set of quantities $(a_1, \ldots, a_n)$ which are under experimental control, which is of the form,*

$$a = f(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n),$$

where $a_1, \ldots, a_k$ have independent physical dimensions. Then there is a function Φ such that the above equation can be rewritten as,

$$\frac{a}{a_1^{p_1} \cdots a_k^{p_k}} = \Phi(\Pi_1, \ldots, \Pi_{n-k}),$$

where $\Pi_1, \ldots, \Pi_{n-k}$ and $\frac{a}{a_1^{p_1} \cdots a_k^{p_k}}$ are dimensionless quantities.

Now we want to apply the Buckingham's Pi theorem to determine how the force of the wind on the barge depends upon the wind velocity. We assume that F can be written as a function of ρ, v, μ and A. We notice that Π_2 is the Reynold's number, $Re = \Pi_2$. Let $\Pi^* := \Pi_1/\Pi_2 = \frac{F}{\rho A v^2}$. Then Re and Π^* are independent dimensionless quantities, and we can apply Buckingham's Pi Theorem. Thus, there is a function Φ such that $\Pi^* = \Phi(Re)$, and consequently,

$$F = \Phi(Re)\rho A v^2.$$

It can be shown experimentally that Φ is approximately constant if $10^2 < Re < 10^5$, see [?]. Thus, if $10^2 < Re < 10^5$, then it holds that

$$F = C\rho A v^2,$$

for a suitable constant C.

(b)

For simplicity, we assume that all sides of the barge have rectangular shape. Then the area on the front is $A_{front} = wh$, and the are on the side is $A_{side} = lh = 5wh = 5A_{front}$. Thus, by part (a), the difference in the force of the wind on the front versus the side of the barge is given by (if $10^2 < Re < 10^5$),

$$F_{front} - F_{side} = C\rho A_{front} v^2 - C\rho A_{side} v^2 = -4C\rho v^2 A_{front} = -4C\rho v^2 wh.$$

By assumption, we have $w = \frac{1}{5}l$ and $h = \frac{1}{10}l$. Thus,the difference in the force of the wind on the front versus the side of the barge is given by (if $10^2 < Re < 10^5$),

$$F_{front} - F_{side} = -4C\rho v^2 wh = -\frac{1}{12}C\rho v^2 l^2.$$

Problem 2. The most common variables in fluid dynamics include:

$$\begin{aligned} l &= \textit{characteristic length scale of the problem} \\ u &= \textit{velocity of flow} \\ \rho &= \textit{density of fluid} \\ \Delta p &= \textit{pressure drop} \\ g &= \textit{gravity} \\ \mu &= \textit{absolute/dynamic viscosity} \end{aligned}$$

(a) Create as many *independent* dimensionless numbers as you can for these variables. Often the form of the dimensionless numbers derived by algebraic calculations using the Buckingham Pi Theorem, or the Rayleigh method, can obscure the physical meaning of the number. For example, the Reynolds number using this approach would appear as $\pi_1 = Re = \rho u l/\mu$. This form hides the physical meaning of Re as the ratio of the internal forces (ρu^2) to the viscous forces ($\mu u/l$). That is, $Re = (internal\ forces)/(viscous\ forces) = (\rho u^2)/(\mu u/L) = \rho u l/\mu$

(b) Identify at least five standard (well known) dimensionless numbers (e.g. see `http://en.wikipedia.org/wiki/Dimensionless_quantity`) that are equivalent to your dimensionless variables. If possible, rewrite your dimensionless numbers so they have a physical meaning, as we did above for the Reynold's number. That is, once you have found the standard dimensionless number that your formula corresponds to, research the physical meaning of that parameter and describe what it means.
Remark: Combinations of dimensionless numbers are also dimensionless numbers. A specific combination of common dimensionless number is often the most appropriate parameter to understand the behavior of a physical system.

Solution.
(a)
The corresponding physical dimensions of the given variables are

$$[g] = LT^{-2}, [\rho] = ML^{-3}, [u] = LT^{-1}, [\mu] = ML^{-1}T^{-1}, [l] = L, [\Delta p] = ML^{-1}T^{-2}.$$

In order to find dimensionless quantities we solve for the null space of the following matrix:

	l	u	ρ	Δp	g	μ
L	1	1	-3	-1	1	-1
T	0	-1	0	-2	-2	-1
M	0	0	1	1	0	1

We find that the null space is spanned by the vectors $(-1, -1, -1, 0, 0, 1)^T$, $(0, -2, -1, 1, 0, 0)^T$ and $(1, -2, 0, 0, 1, 0)^T$. Thus, a complete and independent set of dimensionless quantities is,

$$\Pi_1 = \frac{\mu}{lu\rho},$$
$$\Pi_2 = \frac{\Delta p}{u^2\rho},$$
$$\Pi_3 = \frac{lg}{u^2}.$$

(b)
We identify five standard (well known) dimensionless numbers that are equivalent to the dimensionless variables found in part (a).

- The *Richardson number* $\mathrm{Ri} = \frac{gl}{u^2} = \Pi_3$ is the dimensionless number that expresses the ratio of potential to kinetic energy, i.e.

$$\Pi_3 = \mathrm{Ri} = \frac{\text{potential energy}}{\text{kinetic energy}} = \frac{mgl}{mu^2} = \frac{gl}{u^2}.$$

- The *Euler number* $\mathrm{Eu} = \frac{\Delta p}{\rho u^2} = \Pi_2$ "is a dimensionless number used in fluid flow calculations. It expresses the relationship between a local pressure drop e.g. over a restriction and the kinetic energy per volume, and is used to characterize losses in the flow, where a perfect frictionless flow corresponds to an Euler number of 1" (`http://en.wikipedia.org/wiki/Euler_number_%28physics%29`).

- The *Froude number* $\mathrm{Fr} = \frac{u}{\sqrt{lg}} = \sqrt{\Pi_3^{-1}}$. "In fluid mechanics, the Froude number is used to determine the resistance of a partially submerged object moving through water, and permits the comparison of objects of different sizes." (`http://en.wikipedia.org/wiki/Froude_number`)

- The *Archimedes number* $\mathrm{Ar} = \frac{gl^3\rho_l(\rho-\rho_l)}{\mu^2} = \Pi_3/\Pi_1^2$ "is used to determine the motion of fluids due to density differences." (`http://en.wikipedia.org/wiki/Archimedes_number`)

- The *Dean number* $\mathrm{D} = \frac{\rho V l}{\mu}\left(\frac{l}{2R}\right)^{1/2} = \Pi_1^{-1}\cdot\left(\frac{l}{2R}\right)^{1/2}$ "appears in the so-called Dean Equations. These are an approximation to the full NavierStokes equations for the steady axially uniform flow of a Newtonian fluid in a toroidal pipe, obtained by retaining just the leading order curvature effects." (`http://en.wikipedia.org/wiki/Dean_number`)

2 Phase Plane Analysis

Drawing the Phase Plane

Given $(\dot{x}, \dot{y}) = (f(x), g(y))$ at each point (x, y), we can draw a vector field on the xy-plane. Plot the nullclines, where $\dot{x} = \dot{y} = 0$ and locate the fixed points at their intersections. Along the nullclines, flow is vertical or horizontal.
Different trajectories never intersect, as a unique solution to the system is determined entirely by the initial value.

Poincaré Bendixson Theorem

If a trajectory is confined to a closed, bounded region, and there are no fixed points in the region, then the trajectory must eventually approach a closed orbit.

Classification of Fixed Points

We can write a linear system in the form $\dot{\mathbf{x}} = A\mathbf{x}$. Fixed points occur where $\dot{\mathbf{x}} = \mathbf{0}$. We can classify the behavior at the fixed points by looking at the eigenvalues of A, or, often more simply, by looking at the eigenvalues of the Jacobian matrix at each fixed point.

$$\begin{aligned}
\lambda_1 < \lambda_2 < 0 &\implies \text{stable node} \\
\lambda_1 > \lambda_2 > 0 &\implies \text{unstable node} \\
\lambda_1 < 0 < \lambda_2 &\implies \text{saddle point} \\
\lambda = -\alpha \pm i\beta &\implies \text{stable spiral point} \\
\lambda = \alpha \pm i\beta &\implies \text{unstable spiral point} \\
\lambda = \pm i\omega &\implies \text{elliptic point/center}
\end{aligned}$$

Types of stability:
Attracting - all trajectories that start near x go to x in time
Liapunov stability - all trajectories that start near x remain near x
Neutrally stable - Liapunov stable but not attracting
(Asymptotically) stable - Liapunov stable and attracting

Linearization to Approximate Phase Plane Near Fixed Points

If (x^*, y^*) is a fixed point of the system $\dot{x} = f(x,y), \dot{y} = g(x,y)$, let $(u,v) = (\delta x, \delta y) = (x - x^*, y - y^*)$. Obtain the linearized system

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\ \frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The behavior at the fixed point of this new system is the same as in the original system.

Examples

See Homeworks 3,4, and 5.

Exam Problem

Problem 3 (Phase Plane Analysis). Consider the system of ordinary differential equations

$$\begin{aligned} \dot{x} &= x - 2yx^a, \\ \dot{y} &= -y + xy \end{aligned}$$

that describe a predator-prey system, where x and y are the fractions of the population that are preys or predators. The value a determines how easily the prey x can be captured (and eaten). Consider the cases $a = 0.5$, enter $a = 1$, and $a = 2$.

(a) Find the equilibrium points for the solution,

(b) define the linearized stability equations for perturbations about these equilibrium points, and

(c) analyze their stability by solving for the eigenvalues. If the solution spirals around one of the fix points, then (3d) describe how your can determine the direction of the spiral.

(d) Draw (e.g. pplane.m) the nullclines and the phase plane for the solution and

(e) indicate (classify) the type of each of the equilibrium points.

Solution.
We consider the case $a = 0.5$.
We solve the given equation for fixed points by setting $\dot{x} = 0 = \dot{y}$. The fixed points are $(x,y) = (0,0)$ and $(x,y) = (1, 1/2)$.
We analyze the stability of the fixed points, by looking at the evolution of a small displacement $(\delta x, \delta y)$ about each fixed point.
A linearization of the given equation about $(x,y) = (0,0)$ yields,

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}.$$

Obviously, the corresponding eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. Thus, the fixed point $(x, y) = (0, 0)$ is a hyperbolic point.
A linearization of the given equation about $(x, y) = (1, 1/2)$ yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}1/2 & -2\\ 1/2 & 0\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}. \tag{1}$$

The corresponding characteristic polynomial is $z \mapsto z^2 - 1/2z + 1$, and consequently the corresponding eigenvalues are,

$$\lambda_1 = \frac{1}{4} + \frac{\sqrt{3.75}}{2}i \approx \frac{1}{4} + 0.968246i,$$
$$\lambda_2 = \frac{1}{4} - \frac{\sqrt{3.75}}{2}i \approx \frac{1}{4} - 0.968246i.$$

Thus, $(x, y) = (1, 1/2)$ is an unstable spiral point.
In order to determine the direction of the spiral, we set $\delta y = 0$ and $\delta x > 0$ in the linearization (1). Then we obtain $\dot{\delta y} = \frac{x}{2} > 0$, which implies that y is increasing when y is close to the x-axis and $x > 0$. That is, the motion on the right half-plane is upward, and consequently, the motion of the spiral is anticlockwise.
The phase plane with the nullclines for the solution is shown in figure 1.
Analogously, we see that if $a = 1$ then there is a hyperbolic saddle at $(0, 0)$ and an elliptic point at $(1, 1/2)$, see figure 2.
By the same methods, if $a = 2$ then there is a hyperbolic saddle at $(0, 0)$ and a stable spiral point at $(1, 1/2)$, see figure 3.

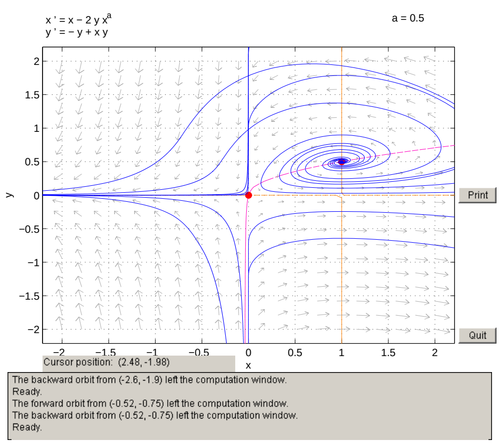

Figure 1: Phase plot of the solution to the ODE of problem 3 with $a = 0.5$. The fixed points are a hyperbolic saddle at $(0, 0)$ and an unstable spiral point at $(1, 1/2)$. The nullclines are shown as pink and orange lines.

Problem 4 (Nonlinear Pendulum). (a) Write the second order differential equation

$$\theta'' + \lambda cos(\theta)\theta' + \omega^2 sin(\theta) = 0$$

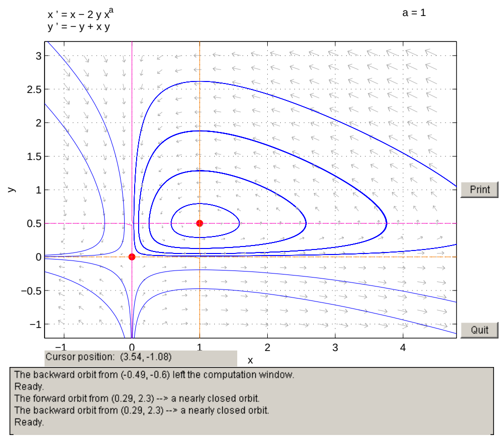

Figure 2: Phase plot of the solution to the ODE of problem 3 with $a = 1$. The fixed points are a hyperbolic saddle at $(0, 0)$ and an elliptic point at $(1, 1/2)$. The nullclines are shown as pink and orange lines.

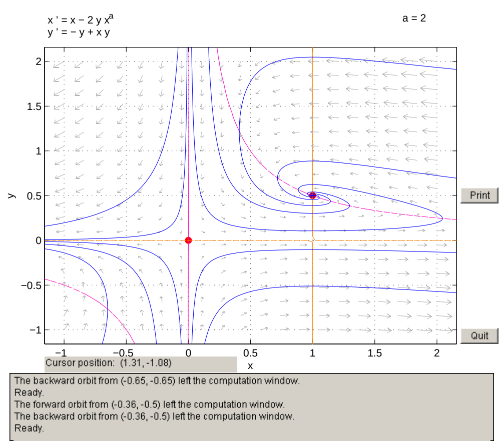

Figure 3: Phase plot of the solution to the ODE of problem 3 with $a = 2$. The fixed points are a hyperbolic saddle at $(0, 0)$ and a stable spiral point at $(1, 1/2)$. The nullclines are shown as pink and orange lines.

as a first order system of equations. Define the frequency as $\omega = 1$, and consider three cases for the friction factor, λ: $\lambda = 0.1$, $\lambda = 0$, and $\lambda = -0.1$.

(b) Analyze the stability of the equilibrium points $(0, 0)$ and $(\pi, 0)$.

(c) Use the phase plane of the solution to describe the difference in the dynamics of the solution as λ changes sign.

Solution.
(a)
We write the given differential equation as a system of two first order differential equations as follows,

$$\begin{cases} \theta' &= \xi, \\ \xi' &= -\lambda \xi \cos\theta - \omega^2 \sin\theta. \end{cases} \tag{2}$$

(b)
Setting $\theta' = 0 = \xi'$ we get that the fixed points are $(\theta, \xi) = (k\pi, 0)$ for all $k \in \mathbb{Z}$.
We analyze the stability of the fixed points $(0, 0)$ and $(\pi, 0)$, by looking at the evolution of a small displacement $(\delta\theta, \delta\xi)$ about each fixed point.
If $(\theta, \xi) = (0, 0)$ then the linearization yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \delta\theta \\ \delta\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\lambda \end{pmatrix}\begin{pmatrix} \delta\theta \\ \delta\xi \end{pmatrix}. \tag{3}$$

The corresponding characteristic polynomial is $z \mapsto z^2 + \lambda z + 1$. We consider three cases for the parameter λ.

$\lambda = 0.1$ In this case the eigenvalues are $-0.0500 \pm 0.09987i$ (approximately, i.e. with the precision of four digits after the decimal dot). Thus, $(0, 0)$ is a stable spiral point. Figure 4 shows a plot of the corresponding phase space.

$\lambda = 0$ In this case the eigenvalues are $\pm i$ (approximately, i.e. with the precision of four digits after the decimal dot). Thus, $(0, 0)$ is an elliptic point. Figure 5 shows a plot of the corresponding phase space.

$\lambda = -0.1$ In this case the eigenvalues are $0.0500 \pm 0.09987i$ (approximately, i.e. with the precision of four digits after the decimal dot). Thus, $(0, 0)$ is an unstable spiral point. Figure 6 shows a plot of the corresponding phase space.

If $(\theta, \xi) = (\pi, 0)$ then the linearization yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \delta\theta \\ \delta\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} \delta\theta \\ \delta\xi \end{pmatrix}. \tag{4}$$

The corresponding characteristic polynomial is $z \mapsto z^2 - \lambda z - 1$. We consider three cases for the parameter λ.

$\lambda = 0.1$ In this case the eigenvalues are 1.0512 and -0.9512 (approximately, i.e. with the precision of four digits after the decimal dot). Thus, $(\pi, 0)$ is a hyperbolic point. Figure 4 shows a plot of the corresponding phase space.

$\lambda = 0$ In this case the eigenvalues are ± 1. Thus, $(\pi, 0)$ is a hyperbolic point. Figure 5 shows a plot of the corresponding phase space.

$\lambda = -0.1$ In this case the eigenvalues are -1.0512 and 0.9512 (approximately, i.e. with the precision of four digits after the decimal dot). Thus, $(\pi, 0)$ is a hyperbolic point. Figure 6 shows a plot of the corresponding phase space.

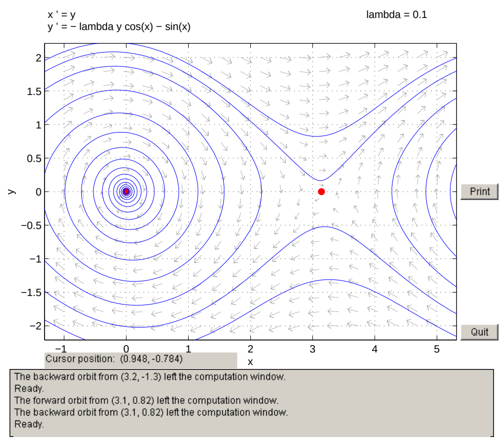

Figure 4: Phase plane of the solution of the ODE given in problem 4 with $\lambda = 0.1$. The fixed point $(\pi, 0)$ is a hyperbolic point. The fixed point $(0, 0)$ is a stable spiral point.

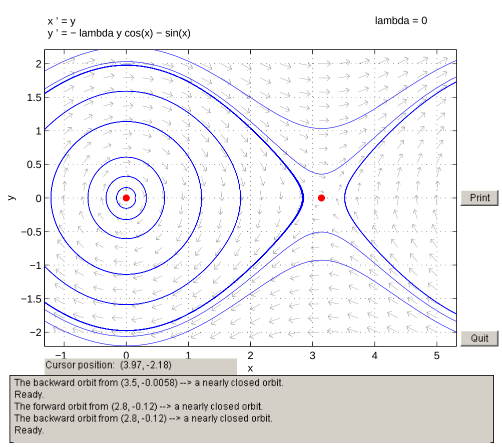

Figure 5: Phase plane of the solution of the ODE given in problem 4 with $\lambda = 0$. The fixed point $(\pi, 0)$ is a hyperbolic point. The fixed point $(0, 0)$ is an elliptic point.

(c)

From figures 4, 5 and 6 we conclude that, as λ changes sign from positive to negative, the fixed point $(0, 0)$ changes from a stable spiral point ($\lambda > 0$) first to an elliptic point ($\lambda = 0$)

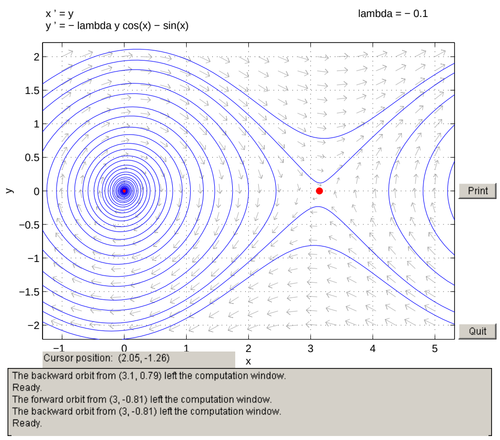

Figure 6: Phase plane of the solution of the ODE given in problem 4 with $\lambda = -0.1$. The fixed point $(\pi, 0)$ is a hyperbolic point. The fixed point $(0, 0)$ is a unstable spiral point.

and then to an unstable spiral point($\lambda < 0$). The same is true for all fixed points of the form $(k\pi, 0)$ with an even k.
The fixed point $(\pi, 0)$ is a hyperbolic point independent of the sign of λ. The same is true for all fixed points of the form $(k\pi, 0)$ with an odd k.

3 Asymptotic Approximations / Perturbation Theory

Asymptotic Approximations

Definitions

Order symbols:

$$f = O(\phi) \iff \lim_{\epsilon \to \epsilon_0} \frac{f(\epsilon)}{\phi(\epsilon)} = L$$

$$f = o(\phi) \iff \lim_{\epsilon \to \epsilon_0} \frac{f(\epsilon)}{\phi(\epsilon)} = 0$$

If $f = \phi + o(\phi)$, then $\phi(\epsilon)$ is an **asymptotic approximation** to $f(\epsilon)$ as $\epsilon \downarrow \epsilon_0$.
A sequence of functions $\phi_m(\epsilon)$ is **well-ordered** as $\epsilon \downarrow \epsilon_0$ iff $\phi_{m+1} = o(\phi_m)$ as $\epsilon \downarrow \epsilon_0$ for all m.
To find asymptotic expansions for a function $f(\epsilon)$, use Taylor's theorem.

Examples

HW 7,8,9

Exam Problems

Problem 5 (Well-Ordered Asymptotic Approximations). (a) Define what it means for a sequence to be *well-ordered*.

(b) Arrange the following sequences so they are well-ordered for small $\varepsilon > 0$: $\phi_1 = \frac{1}{\ln(\varepsilon)}, \phi_2 = \frac{1}{\varepsilon}\sin(\varepsilon^3),\ \phi_3 = e^{-\varepsilon}, \phi_4 = \varepsilon\ln(\varepsilon), \phi_5 = \varepsilon, \phi_6 = \varepsilon^5 e^{-3/\varepsilon}$

(c) Arrange the following sequences so they are well-ordered for small $\varepsilon > 0$: $\phi_1 = 1 + \varepsilon + \frac{\varepsilon^2}{2} - e^\varepsilon, \phi_2 = e^\varepsilon, \phi_3 = e^\varepsilon - 1, \phi_4 = 1 + \varepsilon - e^\varepsilon$

Solution.
(a)
The functions $f_1(\varepsilon), f_2(\varepsilon), \ldots$ are *well ordered* as $\varepsilon \downarrow 0$ if and only if $f_{m+1} = o(f_m)$ as $\varepsilon \downarrow 0$ for all m.
(b)
In the following we use the l'Hospital's Rule for some steps.

- $\lim_{\varepsilon\downarrow 0}\left|\frac{\phi_6}{\phi_2}\right| = \lim_{\varepsilon\downarrow 0}\left|\frac{\varepsilon^5 e^{-3/\varepsilon}}{\frac{1}{\varepsilon}\sin(\varepsilon^3)}\right| = \lim_{\varepsilon\downarrow 0}\left|\frac{\varepsilon^6 e^{-3/\varepsilon}}{\sin(\varepsilon^3)}\right| \leq \lim_{\varepsilon\downarrow 0}\left|\frac{\varepsilon^6}{\sin(\varepsilon^3)}\right| = \lim_{\varepsilon\downarrow 0}\left|\frac{6\varepsilon^5}{\cos(\varepsilon^3)3\varepsilon^2}\right| = 0$
$\Rightarrow \quad \phi_6 = o(\phi_2)$.

- $\lim_{\varepsilon\downarrow 0}\frac{\phi_2}{\phi_5} = \lim_{\varepsilon\downarrow 0}\frac{\frac{1}{\varepsilon}\sin(\varepsilon^3)}{\varepsilon} = \lim_{\varepsilon\downarrow 0}\frac{\sin(\varepsilon^3)}{\varepsilon^2} = \lim_{\varepsilon\downarrow 0}\frac{\cos(\varepsilon^3)3\varepsilon^2}{2\varepsilon} = \lim_{\varepsilon\downarrow 0}\frac{\cos(\varepsilon^3)3\varepsilon}{2} = 0$
  $\Rightarrow \quad \phi_2 = o(\phi_5)$.
- $\lim_{\varepsilon\downarrow 0}\frac{\phi_5}{\phi_4} = \lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\varepsilon\ln(\varepsilon)} = 0 \quad \Rightarrow \quad \phi_5 = o(\phi_4)$.
- $\lim_{\varepsilon\downarrow 0}\frac{\phi_4}{\phi_1} = \lim_{\varepsilon\downarrow 0}\frac{\varepsilon\ln(\varepsilon)}{\frac{1}{\ln(\varepsilon)}} = \lim_{\varepsilon\downarrow 0}\frac{(\ln\varepsilon)^2}{1/\varepsilon} = \lim_{\varepsilon\downarrow 0}\frac{2\ln\varepsilon\frac{1}{\varepsilon}}{-1/\varepsilon^2} = \lim_{\varepsilon\downarrow 0}\frac{2\ln\varepsilon}{-1/\varepsilon} = \lim_{\varepsilon\downarrow 0}\frac{2(1/\varepsilon)}{1/\varepsilon^2} = \lim_{\varepsilon\downarrow 0}\frac{2}{1/\varepsilon} = 0$
  $\Rightarrow \quad \phi_4 = o(\phi_1)$.
- $\lim_{\varepsilon\downarrow 0}\frac{\phi_1}{\phi_3} = \lim_{\varepsilon\downarrow 0}\frac{\frac{1}{\ln(\varepsilon)}}{e^{-\varepsilon}} = \lim_{\varepsilon\downarrow 0}\frac{1}{\ln(\varepsilon)} = 0 \quad \Rightarrow \quad \phi_1 = o(\phi_3)$.

From the above, it follows that the arragement of the given sequence such that it is well ordered is given by

$$\phi_6 << \phi_2 << \phi_5 << \phi_4 << \phi_1 << \phi_3.$$

(c)
In the following we use the l'Hospital's Rule for some steps.

- $\lim_{\varepsilon\downarrow 0}\frac{\phi_1}{\phi_4} = \lim_{\varepsilon\downarrow 0}\frac{1+\varepsilon+\frac{\varepsilon^2}{2}-e^{\varepsilon}}{1+\varepsilon-e^{\varepsilon}} = \lim_{\varepsilon\downarrow 0}\frac{1+\varepsilon-e^{\varepsilon}}{1-e^{\varepsilon}} = \lim_{\varepsilon\downarrow 0}\frac{1-e^{\varepsilon}}{-e^{\varepsilon}} = 0 \quad \Rightarrow \quad \phi_1 = o(\phi_4)$.
- $\lim_{\varepsilon\downarrow 0}\frac{\phi_4}{\phi_3} = \lim_{\varepsilon\downarrow 0}\frac{1+\varepsilon-e^{\varepsilon}}{e^{\varepsilon}-1} = \lim_{\varepsilon\downarrow 0}\frac{1-e^{\varepsilon}}{e^{\varepsilon}} = 0 \quad \Rightarrow \quad \phi_4 = o(\phi_3)$.
- $\lim_{\varepsilon\downarrow 0}\frac{\phi_3}{\phi_2} = \lim_{\varepsilon\downarrow 0}\frac{e^{\varepsilon}-1}{e^{\varepsilon}} = 0 \quad \Rightarrow \quad \phi_3 = o(\phi_2)$.

From the above, it follows that the arragement of the given sequence such that it is well ordered is given by

$$\phi_1 << \phi_4 << \phi_3 << \phi_2.$$

Problem 6 (Asymptotic Approximations). Find the two-term expansion for the solution of the equations as a function of $0 < \varepsilon << 1$.

(a) $x^2 + \varepsilon\sqrt{3 + x^2 - \varepsilon sin(x)} = cos(\varepsilon)$

(b) $\varepsilon^2 x^3 - x + \varepsilon = 0$

Solution.
(a)
We assume that the asymptotic approximation of each root of the given equation can be written as,

$$x = x_0 + \varepsilon x_1.$$

Plugging it into the given equation yields,

$$x_0^2 + 2x_0x_1\varepsilon + \ldots + \varepsilon\sqrt{3 + x_0^2 + \ldots} = 1 + \ldots.$$

The $O(1)$ equation is given by,

$$x_0^2 = 1,$$

and consequently, $x_0 = \pm 1$.
The $O(\varepsilon)$ equation is given by,

$$2x_0x_1 + \sqrt{3 + x_0^2} = 0.$$

Therefore, $x_1 = -\frac{\sqrt{3+x_0^2}}{2x_0}$, where $x_0 = \pm 1$.
Thus, the asymptotic two-term expansions of the two roots of the given equation are,

$$x = 1 - \varepsilon \quad \text{and} \quad x = -1 + \varepsilon.$$

(b)
We want to find the two term asymptotic expansion, for small ε, of each root of the given equation. If $\varepsilon = 0$ then the given equation has only one roots, at $x = 0$. The other roots wander off to $\pm\infty$ as $\varepsilon \downarrow 0$. Thus, we distinguish two cases,

$$x = \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \varepsilon^4 x_4 + \varepsilon^5 x_5 + \ldots, \tag{5}$$
$$x = \varepsilon^\gamma (x_0 + \varepsilon^\alpha x_1). \tag{6}$$

Plugging equation (5) into the given problem, we get,

$$\varepsilon^2(\varepsilon^3 x_1^3 + \ldots) - \varepsilon x_1 - \varepsilon^2 x_2 - \varepsilon^3 x_3 - \varepsilon^4 x_4 - \varepsilon^5 x_5 + \varepsilon = 0.$$

Clearly, the $O(\varepsilon^2)$, $O(\varepsilon^3)$ and $O(\varepsilon^4)$ equations yield that $x_2 = x_3 = x_4 = 0$. The $O(\varepsilon)$ equation is given by $-x_1+1 = 0$, and consequently, $x_1 = 1$. The $O(\varepsilon^5)$ equation is $x_1^3 - x_5 = 0$, and consequently, $x_5 = 1$. Thus, we found the asymptotic expansion of one root to be, $x = \varepsilon + \varepsilon^5$.
Now, plugging equation (6) into the given problem, we get,

$$\varepsilon^{2+3\gamma}(x_0 + \varepsilon^\alpha x_1)^3 - \varepsilon^\gamma x_0 - \varepsilon^{\alpha+\gamma} x_1 + \varepsilon = 0.$$

By a short case distinction, we see that $\gamma = -1$ and $\alpha = 2$. Thus, the above equation becomes,

$$\varepsilon^{-1}(x_0^3 + 3x_0^2 x_1 \varepsilon^2 + \ldots) - \varepsilon^{-1} x_0 - \varepsilon x_1 + \varepsilon = 0.$$

The $O(\varepsilon^{-1})$ equation is $x_0^3 - x_0 = 0$, and consequently $x_0 = 0$ or $x_0 = \pm 1$. But taking $x_0 = 0$ we arrive at the previous solution.
The $O(\varepsilon)$ equation is given by $3x_0^2 x_1 - x_1 + 1 = 0$. Plugging in $x_0 = \pm 1$, it follows that $x_1 = -\frac{1}{2}$.
Thus, we have found the asymptotic two-term expansions of the three roots of the given equation to be,

$$x = \varepsilon + \varepsilon^5 \quad \text{and} \quad x = \pm\varepsilon^{-1} - \frac{1}{2}\varepsilon.$$

Singular Perturbation Problems with Boundary Layers

General Procedure

Step 1: Outer Solution

- Given a boundary value problem, assume that the solution can be expanded in powers of ϵ:

$$y(x) \sim y_0(x) + \epsilon y_1(x) + \cdots$$

- Substitute this approximation into the BVP equation.
- The general (outer) solution is found by solving the ODE made up of $O(1)$ terms. Constants can be found by matching to one or the other given boundary conditions (unless boundary layers occur at both boundaries).

Step 2: Boundary Layer/"Inner Solution"

- Introduce a boundary-layer coordinate:
  For a boundary layer at $x = x_0$, we let the boundary-layer coordinate be $\bar{x} = \frac{x - x_0}{\epsilon^\alpha}$.
- Perform a change of variables to rewrite the BVP in terms of the variable $\bar{x}$.
- Assume that the boundary-layer solution can be expanded as

$$Y(\bar{x}) \sim Y_0(\bar{x}) + \epsilon^\alpha Y_1(\bar{x}) + \cdots$$

- Substitute this approximation into the BVP($\bar{x}$).
- We determine the value for α by matching the orders of terms in the BVP, while maintaining the order of the original equation, i.e., if the first term is $<<$ the second term in the original problem, we maintain that the first term is $<<$ the second term in the boundary-layer problem BVP($\bar{x}$).
- Solve the ODE made up of $O(\frac{1}{\epsilon^\alpha})$ terms. This gives the general inner solution, in the immediate vicinity of x_0. Constants will be determined during Step 3.

Step 3: Matching

- Choose a new variable $\eta = \frac{x}{\epsilon^\beta}$, where $0 < \beta < \alpha$.
- Require for fixed η that

$$\lim_{\epsilon \to 0^+} y(\eta \epsilon^\beta) = \lim_{\epsilon \to 0^+} Y(\eta \epsilon^\beta).$$

  This will allow us to solve for any constants in the inner solution equation.

Step 4: Composite Expansion

- The uniform approximation is given by

$$y_u(x) = y(x) + Y(x) - \lim_{\epsilon \to 0^+} y(\eta \epsilon^\beta).$$

Examples

HW 10,11

Exam Problems

Problem 7 (Singular Perturbation Problem)*.* Using singular perturbation methods, find a uniform approximate solution for the boundary value problem

$$\varepsilon y'' + y' - 2x = 0, \quad y(0) = y(1) = 1, \quad 0 < \varepsilon << 1.$$

Describe, in words, what would change if $-1 << \varepsilon < 0$?

Solution.
We suspect that there is boundary layer at the left end of the interval $[0, 1]$. Then the outer solution fullfils the equation $y' - 2x = 0$ with boundary condition $y(1) = 1$. Thus the outer solution is given by,

$$y_O = x^2. \tag{7}$$

In order to find the inner solution we define, $\xi := \frac{x}{\delta(\varepsilon)}$ and $Y(\xi) := y(\xi\delta(\varepsilon))$. Then the ODE becomes,

$$\frac{\varepsilon}{\delta(\varepsilon)^2} Y'' + \frac{1}{\delta(\varepsilon)} Y' - 2\xi\delta(\varepsilon) = 0.$$

It follows that $\delta(\varepsilon) = \varepsilon$, and the problem that we need to solve is given by,

$$Y'' + Y' = 0, \quad Y(0) = 1.$$

Thus, $Y' + Y = \mathit{const}$, and by separation of variables it follows that

$$\begin{aligned} \int \frac{1}{\mathit{const} - Y} \mathrm{d}Y &= \int 1 \mathrm{d}\xi \\ -\log(\mathit{const} - Y) &= \xi + \mathit{const} \\ \frac{1}{\mathit{const} - Y} &= \mathit{const} \cdot e^{\xi} \\ Y &= c_1 + c_2 e^{-\xi}, \end{aligned}$$

where c_1 and c_2 are constants to be determined. From the boundary condition we get that $c_2 = 1 - c_1$, i.e. $Y = c_1 + (1 - c_1)e^{-\xi}$.
The matching condition states that

$$0 = \lim_{x \to 0} y_O(x) = \lim_{\xi \to \infty} Y(\xi) = c_1. \tag{8}$$

Hence, the inner solution is given by,

$$y_I = e^{-\frac{x}{\varepsilon}}. \tag{9}$$

From equations (7), (9) and (8) it follows that the composite solution is given by,

$$\begin{aligned} y &= y_O + y_I - \lim_{x\to 0} y_O(x) \\ &= x^2 + e^{-\frac{x}{\varepsilon}}. \end{aligned}$$

We check the found approximate solution for correctness by plotting it together with the numerical solution, see figure 7. We see that the approximation is very good for small $\varepsilon > 0$. If $-1 << \varepsilon < 0$, then the sign in front of x/ε in the approximate solution would change from

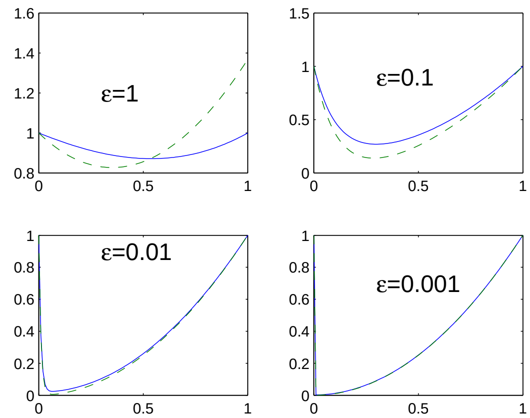

Figure 7: The numerical solution (solid line) and the asymptotic approximation of the exact solution (dashed line) for the boundary value problem 9.

$-$ to $+$, i.e. we would get $y = x^2 + e^{+\frac{x}{\varepsilon}}$.

Problem 8 (Singular Perturbation Problem). Find the composite expansion for the solution of

$$\varepsilon y'' + y^3 + 2y' = 0, \quad y(0) = 0, \quad y(1) = \frac{1}{3}, \quad 0 < \varepsilon << 1.$$

Solution.

The outer solution solves the equation $y_O^3 + 2y_O' = 0$. By separation of variables it follows that

$$\begin{aligned} \int -\frac{2}{y_O^3} \mathrm{d}y_O &= \int 1 \mathrm{d}x \\ (-2)\left(-\frac{1}{2} y_O^{-2}\right) &= x + c \\ y_O^{-1} &= \pm\sqrt{x+c} \\ y_O &= \pm\frac{1}{\sqrt{x+c}}. \end{aligned}$$

We suspect that there is a boundary layer at the left end of the interval $[0, 1]$. Thus, we require that $y_O(1) = \frac{1}{3}$, or equivalently, $\pm\frac{1}{\sqrt{1+c}} = \frac{1}{3}$. Hence, the outer solution is given by,

$$y_O = \frac{1}{\sqrt{x+8}}. \tag{10}$$

In order to find the inner solution we define, $\xi := \frac{x}{\delta(\varepsilon)}$ and $Y(\xi) := y(\xi\delta(\varepsilon))$. Then the ODE becomes,

$$\frac{\varepsilon}{\delta(\varepsilon)^2}Y'' + Y^3 + \frac{2}{\delta(\varepsilon)}Y' = 0.$$

It follows that $\delta(\varepsilon) = \varepsilon$, and the problem that we need to solve is given by,

$$Y'' + 2Y' = 0, \quad Y(0) = 0.$$

Thus, $Y' + 2Y = c_1'$ (c_1' is a suitable constant), and by separation of variables it follows that

$$\begin{aligned}
\int \frac{1}{c_1' - 2Y}\mathrm{d}Y &= \int 1\mathrm{d}\xi \\
\left(-\frac{1}{2}\right)\log(c_1' - 2Y) &= \xi + const \\
(c_1' - 2Y)^{-1/2} &= const \cdot e^{\xi} \\
(c_1' - 2Y)^{-1} &= const \cdot e^{2\xi} \\
c_1' - 2Y &= \frac{1}{const \cdot e^{2\xi}} \\
Y &= c_1 + c_2 e^{-2\xi},
\end{aligned}$$

where c_1 and c_2 are constants to be determined. By the boundary condition, $Y(0) = 0$, it follows that $c_1 = -c_2$, i.e. $Y = c_1(1 - e^{-2\xi})$.
The matching condition states that

$$\frac{1}{\sqrt{8}} = \lim_{x \to 0} y_O(x) = \lim_{\xi \to \infty} Y(\xi) = c_1. \tag{11}$$

Hence, the inner solution is given by,

$$y_I = \frac{1}{\sqrt{8}}(1 - e^{-2\frac{x}{\varepsilon}}). \tag{12}$$

From equations (10), (12) and (11) it follows that the composite solution is given by,

$$\begin{aligned}
y &= y_O + y_I - \lim_{x \to 0} y_O(x) \\
&= \frac{1}{\sqrt{x+8}} - \frac{e^{-2\frac{x}{\varepsilon}}}{\sqrt{8}}.
\end{aligned}$$

We check the found approximate solution for correctness by plotting it together with the numerical solution, see figure 8. We see that the approximation is very good for small $\varepsilon > 0$.

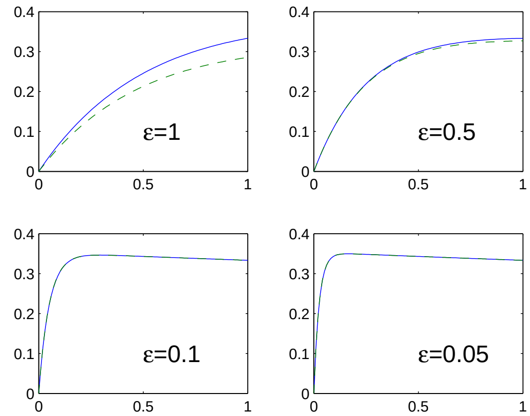

Figure 8: The numerical solution (solid line) and the asymptotic approximation of the exact solution (dashed line) for the boundary value problem 10.

Part II

Applied Mathematics 2

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4 Fourier Analysis

Fourier Transform

Convolution $(u * v)(x) = \int_{\mathbb{R}} u(x-y)v(y)\mathrm{d}y = \int_{\mathbb{R}} u(y)v(x-y)\mathrm{d}y$.

Fourier Transform $\hat{u}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} u(x)\mathrm{d}x$.

Inverse Fourier Transform $u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\hat{u}(\xi)\mathrm{d}\xi$.

Some Properties of the Fourier Transform :

- Parseval's equality: $\|\hat{u}\| = \sqrt{2\pi}\|u\|$.
- $\widehat{u * v}(\xi) = \hat{u}(\xi)\hat{v}(\xi)$.
- $\widehat{u+v}(\xi) = \hat{u}(\xi) + \hat{v}(\xi), \hat{cu}(\xi) = c\hat{u}(\xi)$.
- $\widehat{(u_x)}(\xi) = i\xi\hat{u}(\xi)$.

Solving a PDE with Fourier Analysis

Steps to solve a PDE of the form $u_t = Au + Bu_x + Cu_{xx} + \ldots + Nu_{x\ldots x}$, where $x \in [-L/2, L/2]$.

(a) By the IFT,

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\frac{\partial}{\partial t}\hat{u}(\xi,t)\mathrm{d}\xi &= u_t = Au + Bu_x + Cu_{xx} + \ldots + Nu_{x\ldots x} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}(A\hat{u}(\xi,t) + i\xi B\hat{u}(\xi,t) + (i\xi)^2C\hat{u}(\xi,t) + \ldots + (i\xi)^n N\hat{u}(\xi,t))\mathrm{d}\xi.\end{aligned}$$

(b) Removing the $\int$-symbols yields the ODE,

$$\frac{\partial}{\partial t}\hat{u}(\xi,t) = (A + i\xi B + (i\xi)^2C + \ldots + (i\xi)^n N)\hat{u}(\xi,t).$$

(c) The solution of the ODE is given by,

$$\hat{u}(\xi,t) = e^{(A+i\xi B+(i\xi)^2C+\ldots+(i\xi)^n N)t}\hat{u}(\xi,0).$$

(d) By the IFT,

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi(x+Bt+\ldots)}e^{t(A-\xi^2C+\ldots)}\hat{u}(\xi,0)\mathrm{d}\xi.$$

(e) Assume that $u(x,0) = \sum_{k=0}^{\infty} a_k\cos(kx) + b_k\sin(kx)$, then

$$u(x,t) = \sum_{k=0}^{\infty} e^{t\left(A-\left(\frac{2\pi}{L}k\right)^2C+\ldots\right)}\left[a_k\cos\left(\frac{2\pi}{L}k(x+Bt+\ldots)\right) + b_k\sin\left(\frac{2\pi}{L}k(x+Bt+\ldots)\right)\right].$$

Discrete Fourier Transform

Definition (DFT& IDFT)**.** For $k \in \left\{-\frac{N}{2}+1, \ldots, \frac{N}{2}\right\}$ the DFT is a sequence defined by,

$$F_k = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f(x_n) \exp(-i2\pi nk/N).$$

The inverse DFT is a sequence defined by,

$$f_n = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} F_k \exp(i2\pi nk/N),$$

for all $n \in \left\{-\frac{N}{2}+1, \ldots, \frac{N}{2}\right\}$.

Some Properties of the DFT:

- If we assume that the inverse DFT is true not only for grid points of the form $\frac{nA}{N}$, but at any point in $x \in (-A/2, A/2)$ then differentiation with respect to x yields,

  $$f'(x) = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} F_k \frac{i2\pi k}{A} \exp\left(\frac{i2\pi k}{A} x\right).$$

  Thus, we conclude that the derivative of f at the grid points $\frac{nA}{N}$ ($n \in \left\{-\frac{N}{2}+1, \ldots, \frac{N}{2}\right\}$) can be estimated by,

  $$f'(x_n) = f'_n = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} F_k \frac{i2\pi k}{A} \exp\left(\frac{i2\pi kn}{N}\right).$$

- *Reciprocity Relations*: $A\Omega = N$, $\Delta x \Delta \omega = \frac{1}{N}$, where N is the number of grid points, $x \in (-A/2, A/2)$, $\omega \in (-\Omega/2, \Omega/2)$.
- Largest period $= A \quad \Rightarrow \quad$ Lowest frequency $= \frac{1}{A}$,
  Shortest period $= 2\Delta x = 2\frac{A}{N} \quad \Rightarrow \quad$ Highest frequency $= \frac{N}{2A}$.
- The N modes of the DFT is the set of functions $e^{i2\pi\omega_k x_n}$, where $\omega_k = \frac{k}{A}$ and $x_n = \frac{nA}{N}$.

5 Heat Equation

Derivation (in class midterm problem 1(a))

We derive the heat equation for a rod assuming constant thermal properties with variable cross-sectional area $A(x)$ and sources $Q(x)$ by considering the total thermal energy between $x = a$ and $x = b$.
Let $e(x,t)$ be the thermal energy density. Then the total thermal energy between $x = a$ and $x = b$ is given by $\int_a^b e(x,t)A(x)\mathrm{d}x$.
Let $\phi(x,t)$ denote the heat flux. Then the conservation of heat energy equation is given by,

$$\begin{aligned}\frac{\partial}{\partial t}\int_a^b e(x,t)A(x)\mathrm{d}x &= \phi(a,t)A(a) - \phi(b,t)A(b) + \int_a^b Q(x)A(x)\mathrm{d}x \\ &= -\int_a^b \frac{\partial}{\partial x}(\phi(x,t)A(x))\mathrm{d}x + \int_a^b Q(x)A(x)\mathrm{d}x.\end{aligned}$$

Since the above equation is true for any choice of $a < b$ in the domain, and because of $\frac{\partial}{\partial t}\int_a^b e(x,t)A(x)\mathrm{d}x = \int_a^b \frac{\partial}{\partial t}e(x,t)A(x)\mathrm{d}x$, it follows that

$$\frac{\partial}{\partial t}(e(x,t)A(x)) = -\frac{\partial}{\partial x}(\phi(x,t)A(x)) + Q(x)A(x).$$

Let c denote the specific heat, and let ρ be the mass density. By assumption c and ρ are constant. For a thin slice $[x, x+\Delta x]$ the total heat energy can be approximated as $e(x,t)\int_x^{x+\Delta x} A(x)\mathrm{d}x$ and it can also be approximated as $c\rho u(x,t)\int_x^{x+\Delta x} A(x)\mathrm{d}x$. Thus, it follows that

$$e(x,t) = c\rho u(x,t).$$

Plugging this into the above equation yields,

$$c\rho\frac{\partial}{\partial t}(u(x,t)A(x)) = -\frac{\partial}{\partial x}(\phi(x,t)A(x)) + Q(x)A(x).$$

Now, by Fourier's law of heat conduction $\phi = -K_0 u_x$ ($K_0 > 0$ constant) it follows that

$$c\rho\frac{\partial}{\partial t}(u(x,t)A(x)) = K_0\frac{\partial}{\partial x}(u_x(x,t)A(x)) + Q(x)A(x).$$

Consequently, the heat equation is given by,

$$(Au)_t = k(Au_x)_x + \frac{k}{K_0}QA,$$

where $k = \frac{K_0}{c\rho}$ is the thermal diffusivity.

Boundary Conditions

Prescribed Temperature $u(0,t) = u_B(t)$.

Prescribed Heat Flux / Insulated Boundary $-K_0(0)u_x(0,t) = \phi(t)$,

Newton's Law of Cooling $-K_0(0)u_x(0,t) = -H(u(0,t) - u_B(t))$, where $H > 0$ is the *heat transfer coefficient* and constant (no minus sign in front of H at the right boundary R, i.e. $-K_0(R)u_x(R,t) = H(u(R,t) - u_B(t))$).

One boundary condition occurs at each boundary. It is not necessary that both boundaries satisfy the same kind of boundary condition.

Maximum Principle

Theorem (Weak Maximum Principle)**.** *If $u(x,t)$ satisfies the diffusion equation $u_t = ku_{xx}$ in a rectangle $(x,t) \in [0,l] \times [0,T]$, then the maximum value of u is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$).*

Theorem (Strong Maximum Principle)**.** *The maximum cannot be assumed on the interior but only on the bottom or the lateral sides of the rectangle (unless u is constant).*

Uniqueness Proofs

Assume two solutions u and v. Then $w := u - v$ is also a solution. Then,

- either apply the maximum principle to w,
- or apply the *energy method*, that is, consider $0 = 0 \cdot w = \text{PDE} \cdot w =$ "sum of some terms differentiated w.r.t x", then integrate w.r.t. x, then prove that $\int w^2 \leq 0$ or something similar.

6 Method of Characteristics

From P.Lax "The formation and decay of shock waves"

Characteristic Curves, Signal Speed. Given $u_t + (f(u))_x = 0$, denote $\frac{\partial f}{\partial u} = a(u)$, then $u_t + a(u)u_x = 0$.
Since $\frac{\partial u}{\partial t} = u_t + \frac{\partial x}{\partial t}u_x$ (so, $\frac{\partial x}{\partial t} = a(u) \Rightarrow \frac{\partial u}{\partial t} = 0$), it follows that u is constant along trajectories $x(t)$ which propagate with speed $a(u)$ (*signal speed*). Such trajectories are called *characteristics*. The value of u along the characteristic can be determined from the initial values $u(x, 0)$.

Weak Solution. u is a weak solution ($= u$ satisfies the PDE in the sense of distributions), if for every continuously differentiable ϕ with compact support it holds that $\int\int \phi_t u + \phi_x f(u)\mathrm{d}x\mathrm{d}t = 0$.

Rankine-Hugoniot Jump Condition. When the characteristics cross there exists a unique weak solution which has a jump/shock. The shock occurs along the line y whose slope is given by $\frac{\partial y}{\partial t} = s = \frac{f(u_r)-f(u_l)}{u_r-u_l}$.

Lax Entropy Condition. A weak solution with a shock along the line with slope $s = \frac{f(u_r)-f(u_l)}{u_r-u_l}$ exists and is unique if and only if $a(u_l) > a > a(u_r)$.

From L. Escauriaza "Method Of Characteristics"

Consider a first-order quasilinear equation of a function of two variables with data prescribed on a curve Γ in the xy-plane,

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u),$$
$$u|_\Gamma = \phi.$$

Derivation of the method of characteristics

The normal to the solution surface $S = \{(x, y, u(x, y))\}$ at the point $(x, y, u(x, y))$ is given by $N(x, y) = (u_x, u_y, -1)$. If u is a solution then at each point (x, y) we have $(a, b, c) \cdot (u_x, u_y, -1) = 0$. Thus, the vector $(a(x, y, u), b(x, y, u), c(x, y, u))$ lies in the tangent plane to S. Therefore, in order to find the solution u, we need to construct a surface S, such that at each point (x, y, z) of S, the vector $(a(x, y, z), b(x, y, z), c(x, y, z))$ lies in the tangent plane. Thus, we construct curves which lie in S, the *characteristic curves*. This reduces the PDE to a system of ODEs. Then S is the union of all characteristic curves.

The method of characteristics

We parametrize $\Gamma = \{(\gamma_1(r), \gamma_2(r))\}$. A characteristic curve $C = \{(x, y, z)\}$ satisfies the following system of ODEs,

$$\frac{\partial x}{\partial s} = a(x, y, z),$$
$$\frac{\partial y}{\partial s} = b(x, y, z),$$
$$\frac{\partial z}{\partial s} = c(x, y, z),$$

with initial conditions,

$$x(r, 0) = \gamma_1(r),$$
$$y(r, 0) = \gamma_2(r),$$
$$z(r, 0) = \phi(r).$$

Solve the system using ODE theory. Then, $u(x, y) = z(r(x, y), s(x, y))$ solves the PDE.
A solution exists as long as Γ is a **noncharacteristic boundary**, that is, as long as

$$(a(\gamma_1(r), \gamma_2(r), \phi(r)), b(\gamma_1(r), \gamma_2(r), \phi(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) \neq 0.$$

7 Linearization Of PDEs

[Based on R. Leveque "Conservation Laws and Differential Equations".]

Nonlinear Equations in Fluid Dynamics

Consider gas with variable density ρ (i.e. compressible), velocity u, pressure p, and *isentropic* flow (i.e. $p = P(\rho)$ with $P'(\rho) > 0$ for $\rho > 0$).

$$\rho_t + (\rho u)_x = 0 \,\text{(conservation of mass)},$$
$$(\rho u)_t + (\rho u^2 + P(\rho))_x = 0 \,\text{(conservation of momentum)}.$$

Let $q = (\rho, \rho u)^T = (q^{(1)}, q^{(2)})^T$ and $f(q) = (q^{(2)}, (q^{(2)})^2/q^{(1)} + P(q^{(1)}))^T = (\rho u, \rho u^2 + P(\rho))^T$. Then the system can be written as,

$$q_t + f'(q)q_x = 0,$$

and differentiation yields,

$$f'(q) = \begin{pmatrix} 0 & 1 \\ -u^2 + P'(\rho) & 2u \end{pmatrix}.$$

We write,

$$\begin{aligned} q(x,t) &= q_0 + \tilde{q}(x,t), \\ u &= u_0 + \tilde{u}, \\ \rho &= \rho_0 + \tilde{\rho}, \end{aligned}$$
$$P(\rho) = p = p_0 + \tilde{p} = P(\rho_0 + \tilde{\rho}) = P(\rho_0) + P'(\rho_0)\tilde{\rho} + \ldots \text{ and } p_0 = P(\rho_0) \quad \Rightarrow \quad \tilde{p} \approx P'(\rho_0)\tilde{\rho},$$
$$\rho_0 u_0 + \rho_0 \tilde{u} + \tilde{\rho} u_0 + \ldots = (\rho_0 + \tilde{\rho})(u_0 + \tilde{u}) = \rho u = \rho_0 u_0 + \widetilde{(\rho u)} \quad \Rightarrow \quad \widetilde{(\rho u)} \approx \rho_0 \tilde{u} + \tilde{\rho} u_0.$$

With the aid of the above formulas, the linearized equation $q_t + f'(q_0)q_x = 0$ can be written out in terms of its components as the system,

$$\tilde{p}_t + u_0 \tilde{p}_x + K_0 \tilde{u}_x = 0,$$
$$\rho_0 \tilde{u}_t + \tilde{p}_x + \rho_0 u_0 \tilde{u}_x = 0,$$

where $K_0 = \rho_0 P'(\rho_0)$.
Equivalently, the linearized equation can be written in the form $q_t + Aq_x = 0$ with $q = (p, u)$ (abuse of notation: we dropped the tilde and re-defined q).

Sound Waves

We expect that the general solution is a superposition of waves moving in each direction, with a constant speed s (the speed of sound) with its shape unchanged. This suggests looking for solutions of the form $q(x,t) = \bar{q}(x - st)$, where $\bar{q}$ is some function of one variable. By basic manipulation it follows that $A\bar{q}'(x - st) = s\bar{q}'(x - st)$ (or $= s\bar{q}'(x + st)$). That is, s is an eigenvalue of A and $\bar{q}$ is the corresponding eigenvector.

Hyperbolicity Of Linear Systems

Definition. A linear system of the form $q_t + Aq_x = 0$ is called *hyperbolic* if the matrix A is diagonalizable with real eigenvalues.

We denote the EV $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ and the EVec $r_1, \ldots, r_m$. Let $R = [r_1|\ldots|r_m]$, then $R^{-1}AR = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$. It follows that

$$w_t + \Lambda w_x = 0,$$

where $w(x,t) = R^{-1}q(x,t)$.
The solution will consist of m waves travelling at the *characteristic speeds* $\lambda_1, \ldots, \lambda_m$.

8 Classification of PDE's

The most general form of a linear, second order PDE in two independent variables x, y and the dependent variable $u(x, y)$ is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0,$$

with constant $A, \ldots, G$. This equation is called *elliptic* if $B^2 - 4AC < 0$, *parabolic* if $B^2 - 4AC = 0$, and *hyperbolic* if $B^2 - 4AC > 0$.

Hyperbolic PDE	Parabolic PDE	Elliptic PDE
Wave Eq.	Heat / Diffusion Eq.	Laplace / Poisson Eq.
$u_{tt} = u_{xx}$	$u_t = u_{xx}$	$-\Delta u = 0$
IVP, or IBVP+BC	IVP, or IBVP	BVP
Euler Eq.	Navier Stokes	Stokes

9 Green's Functions for One-Dimensional Equations

Definition (Singularity functions)**.** A *singularity function* $K(x,\xi)$ of the operator $\mathcal{L}$ defined by $\mathcal{L}u(x) = -u''(x) - c(x)u(x)$ is characterized by three properties:

(a) K is continuous,

(b) K_x is continuous in $x < \xi$ and in $x > \xi$, and $K_x(x^+, x) - K_x(x^-, x) = -1$,

(c) K_{xx} is continuous and $\mathcal{L}K = 0$ for $x \neq \xi$.

If K is a singularity function then so is $K + H$, where $H(x,\xi)$ is any function with continuous H and H_x and with $\mathcal{L}H = 0$.

Definition (Green's function)**.** The *Green's function* $G(x,\xi)$ for the operator $\mathcal{L}$ and the domain (a,b) with Dirichlet boundary conditions is the singularity function that satisfies the homogeneous Dirichlet condition $G(a,\xi) = 0$ and $G(b,\xi) = 0$.

Theorem. *If u satisfies $u'' + cu = -f$ on (a,b) with BC $u(a) = 0 = u(b)$, then*

$$u(\xi) = \int_a^b G(x,\xi) f(x) \mathrm{d}x,$$

for all $\xi \in (a,b)$.

Theorem. *The solution of $u'' + cu = 0$ with BC $u(a) = h_0$ and $u(b) = h_1$ satisfies*

$$u(\xi) = G_x(a,\xi)h_0 - G_x(b,\xi)h_1,$$

for all $\xi \in (a,b)$.

Theorem (Reciprocity Principle)**.** $G(x,\xi) = G(\xi,x)$ *for all $x, \xi \in (a,b)$.*

Finding the Green's function and using it to solve a PDE

A singularity function of $-\frac{\partial^2}{\partial x^2}$ is given by

$$K(x,\xi) = -\frac{1}{2}|x - \xi|.$$

The Green's function for the interval $(0,1)$ can be found by solving $H_{xx} = 0$ with boundary conditions $H(0,\xi) = -K(0,\xi)$ and $H(1,\xi) = -K(1,\xi)$, and then setting $G = K + H$. This yields

$$G(x,\xi) = -\frac{1}{2}|x - \xi| + \frac{1}{2}(x + \xi) - x\xi.$$

By the above theorems, it holds that $u(x) = \int_a^b G(x,\xi)f(\xi)\mathrm{d}\xi$ and $u(x) = G_\xi(x,a)h_0 - G_\xi(x,b)h_1$. Thus, the solution of $-u'' = f$ with BC $u(0) = h_0$ and $u(1) = h_1$ is given by,

$$
\begin{aligned}
u(x) &= \int_0^1 G(x,\xi)f(\xi)\mathrm{d}\xi + G_\xi(x,0)h_0 - G_\xi(x,1)h_1 \\
&= (1-x)\int_0^x \xi f(\xi)\mathrm{d}\xi + x\int_x^1 (1-\xi)f(\xi)\mathrm{d}\xi + (1-x)h_0 + xh_1.
\end{aligned}
$$